# The Contravariant Metric Approach to General Relativity

John Stachel
Center for Einstein Studies, BU

The Renaissance of General Relativity
Berlin, 5 December 2015

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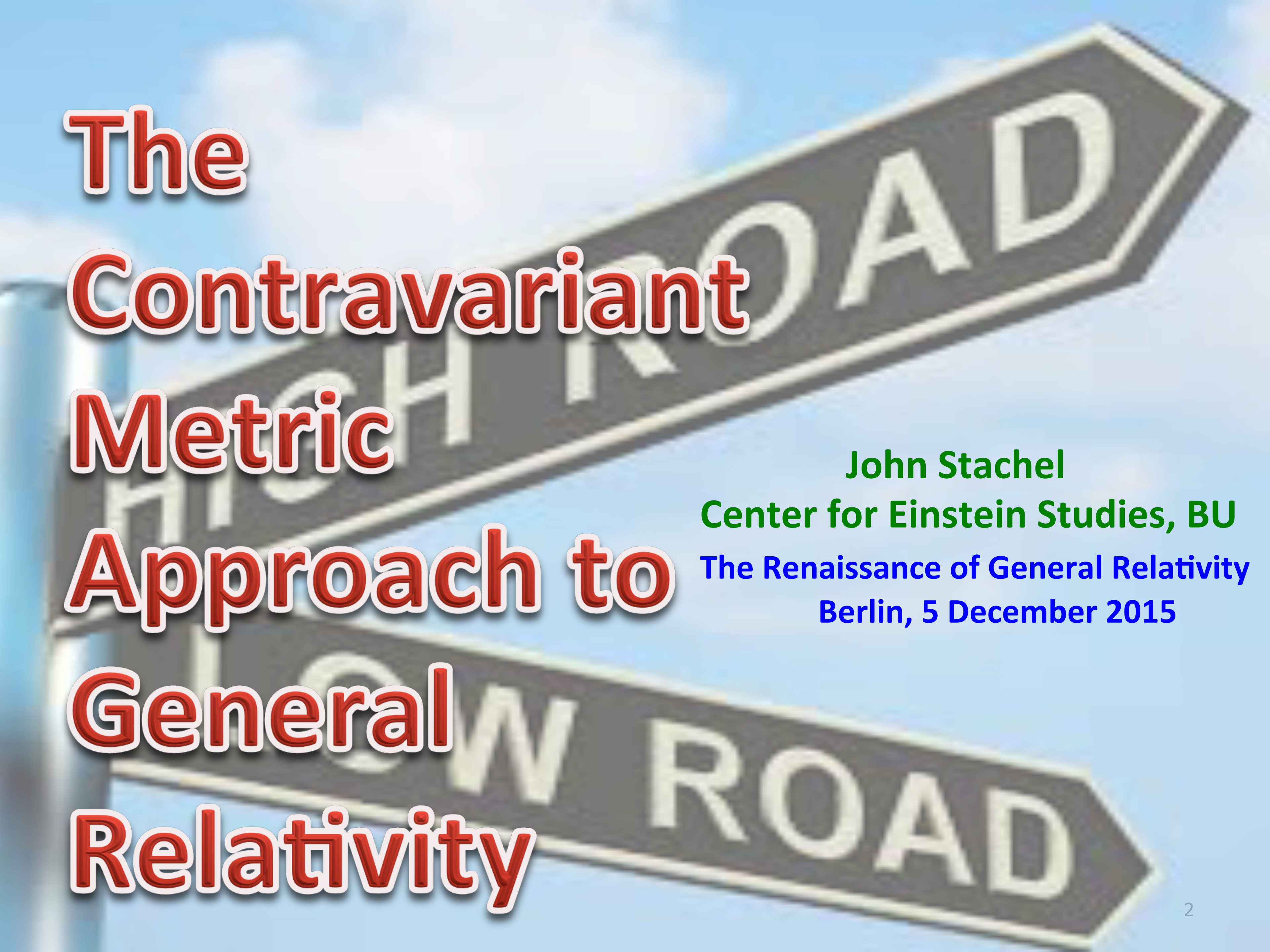


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# Contravariant Metric

We shall take the contravariant metric $g^{\mu\nu}$ and covariant vectors $\omega_\mu$ as the starting point for discussion of the pseudo-metric. Traditionally, of course, just the opposite is done: one starts with the covariant metric $g_{\mu\nu}$ and contravariant vectors $V^\mu$.

But the physical interpretation of the first-order formalism for general relativity suggests taking the contravariant approach.

# First-Order Lagrangian

**The Lagrangian density is:**

$$\mathcal{L} = (-g_b)^{1/2} g^{\mu\nu} A_{\mu\nu}(\Gamma, \partial\Gamma)$$

$$= (-g^{\#})^{-1/2} g^{\mu\nu} A_{\mu\nu}(\Gamma, \partial\Gamma).$$

**In this equation, $A_{\mu\nu}(\Gamma, \partial\Gamma)$ is the affine Ricci tensor written as a function of the symmetric affine connection $\Gamma$ and its first derivatives $\partial\Gamma$**

**Thus, only the contravariant metric is needed to derive the gravitational field equations, as we shall discuss in detail later.**


Take the high road
the view is better

# So We'll Take the High Road!

# My Apologies in Advance

Time limits require brevity and brevity is the mother of dogmatism.

None of my statements should be interpreted dogmatically– they are meant to stimulate critical thinking and further discussion.

# "Flat" *versus* "Sharp"

Quantities formed from the covariant metric $g_{\mu\nu}$ will be called "flat" & distinguished by the symbol "$_b$"; while quantities formed from the contravariant metric $g^{\mu\nu}$ will be called "sharp" & distinguished by the symbol "#".

# Space-Time Structures in GR

**Zeroth order**: **4-Dim Differentiable Manifold** $X_4$

***Diff*** **(**$X_4$**), Unimodular** ***SDiff(***$X_4$***)*****, Tangent and Cotangent Spaces, Volume elements**

**First Order**: **Affine Connections, Pseudo-metrics and Connections, Compatibility conditions**

**Conformal and Projective structures, Torsion**

**Second order**: **Affine, Metric, Conformal and Projective Curvature tensors and a**

**Ricci tensor for each curvature tensor**

# Space-Time Structures in GR

I shall emphasize certain relations between these structures:

Compatibility between Two

Co-Determination of a Third

NB: All structures are four-dimensional unless otherwise indicated

(e.g., "volume" = "four-volume")

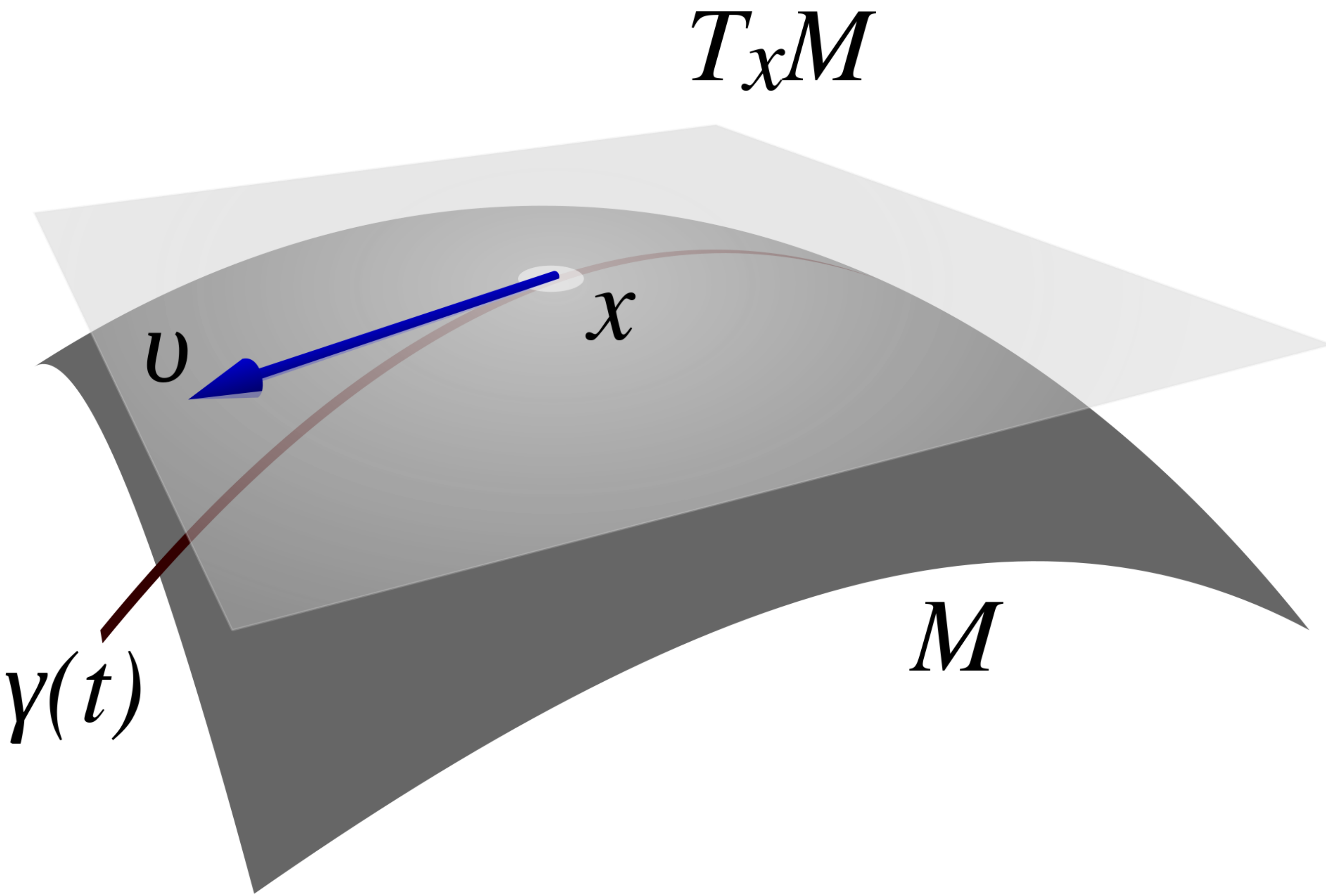
"Metric" = "pseudo-metric with Lorentz signature (+ - - -)"

# Differentiable Manifold: Zero$^{th}$ Order Structures

4-Dim **Differentiable Manifold** $X_4$
with **Symmetry Group**: $Diff(X_4)$

Tangent Space $T_p(X_4)$ at each $p \varepsilon X_4$
**Centered Affine Space** with
Symmetry Group: $GL(4, R)$

# Tangent Space



# The General Linear Group and its Subgroups

$G_a$ The affine group $\Delta \neq 0$, $E_n$ Affine Space

$G_{ho}$ Homogeneous trans'ns $\Delta \neq 0$, Centered $E_n$

$G_{eq}$ Equivoluminar $\Delta = \pm 1$, $E_n$ with given unit volume

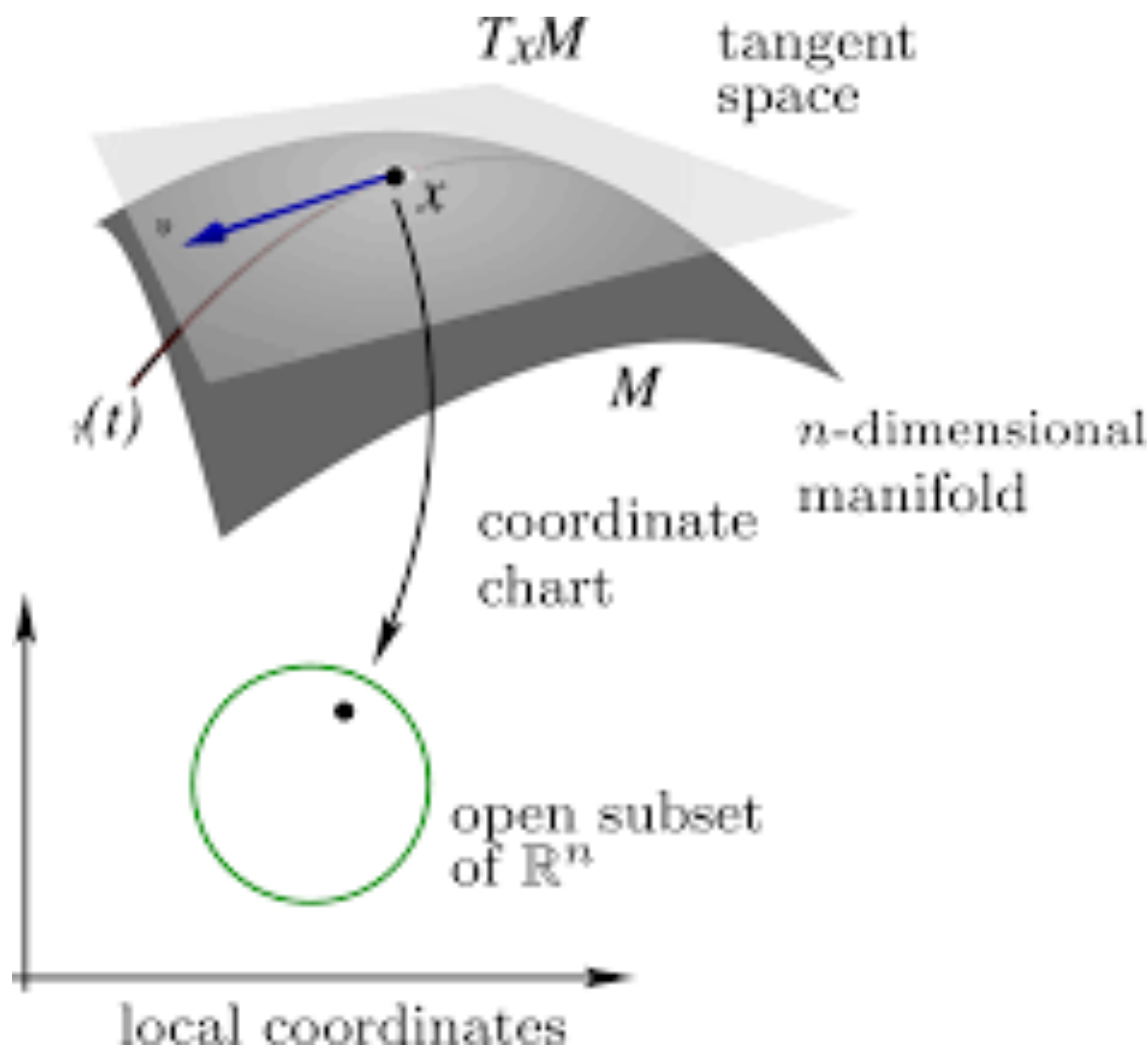
$G_{ea}$ Special affine trans'ns $\Delta = +1$, $E_n$ with given unit volume and screw sense

$G_{pso}$ Pseudo-orthogonal trans'ns $\Delta \neq 0$, $C_n$ (rotations, reflections and similarity transformations)

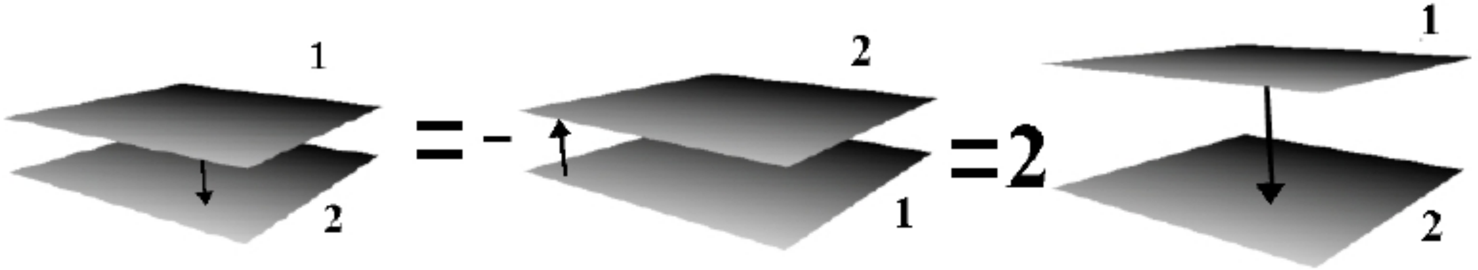
$G_{pson}$ Pseudorthonormal trans'ns $\Delta = \pm 1$, $R_n$ (rotations and reflections)

$G_{psor}$ Pseudo-rotations $\Delta = 1$, Oriented $R_n$ (rotations)

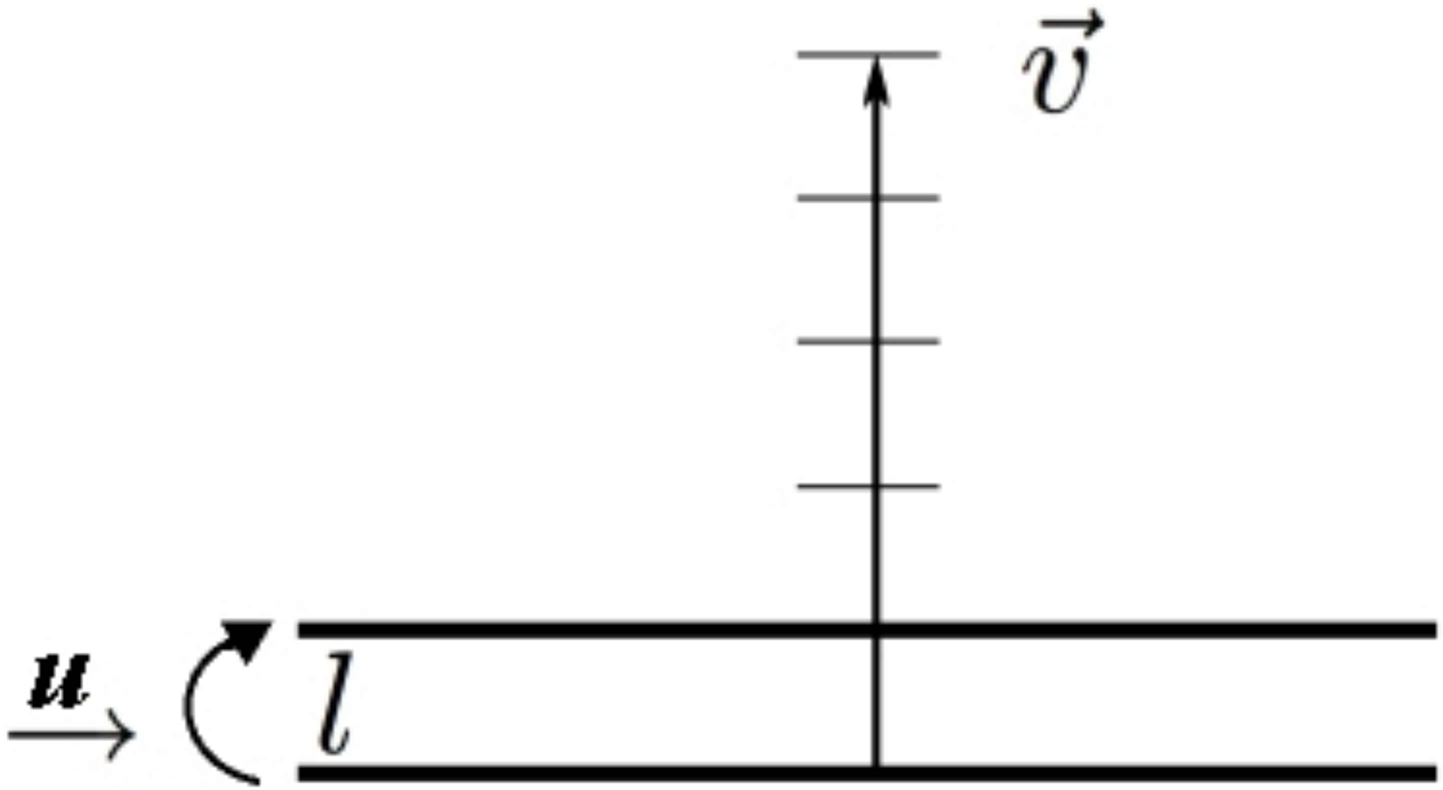
# Coordinatization



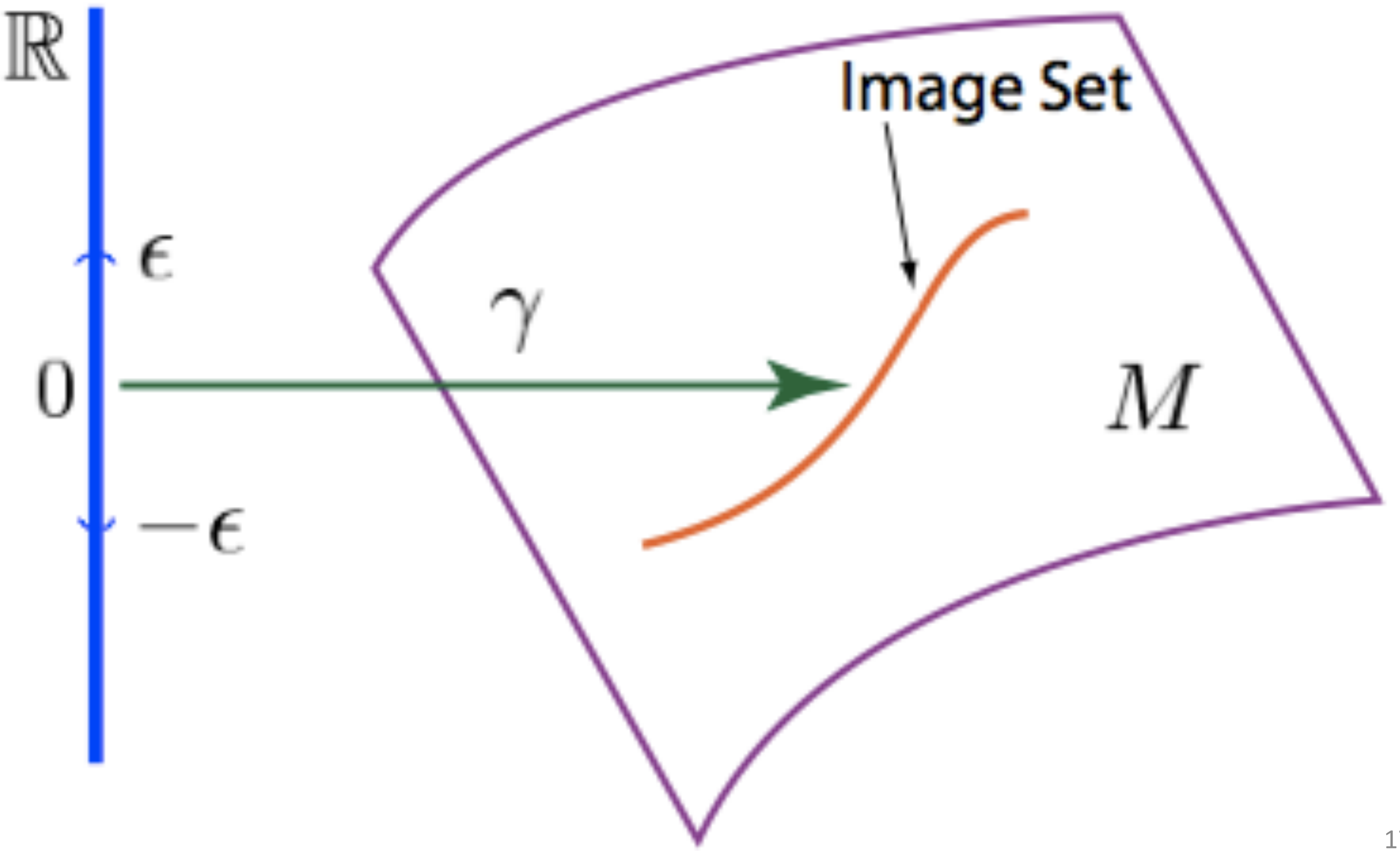
# Vectors and Covectors



# Contraction of vector and covector



# A Curve on the Manifold *M* is a smooth map *γ*: -ε<*R*<ε⟶*M*



# Zero<sup>th</sup> Order Dualities

## Differentiable Manifolds

**Vectors $V$** **Covectors $\omega$**

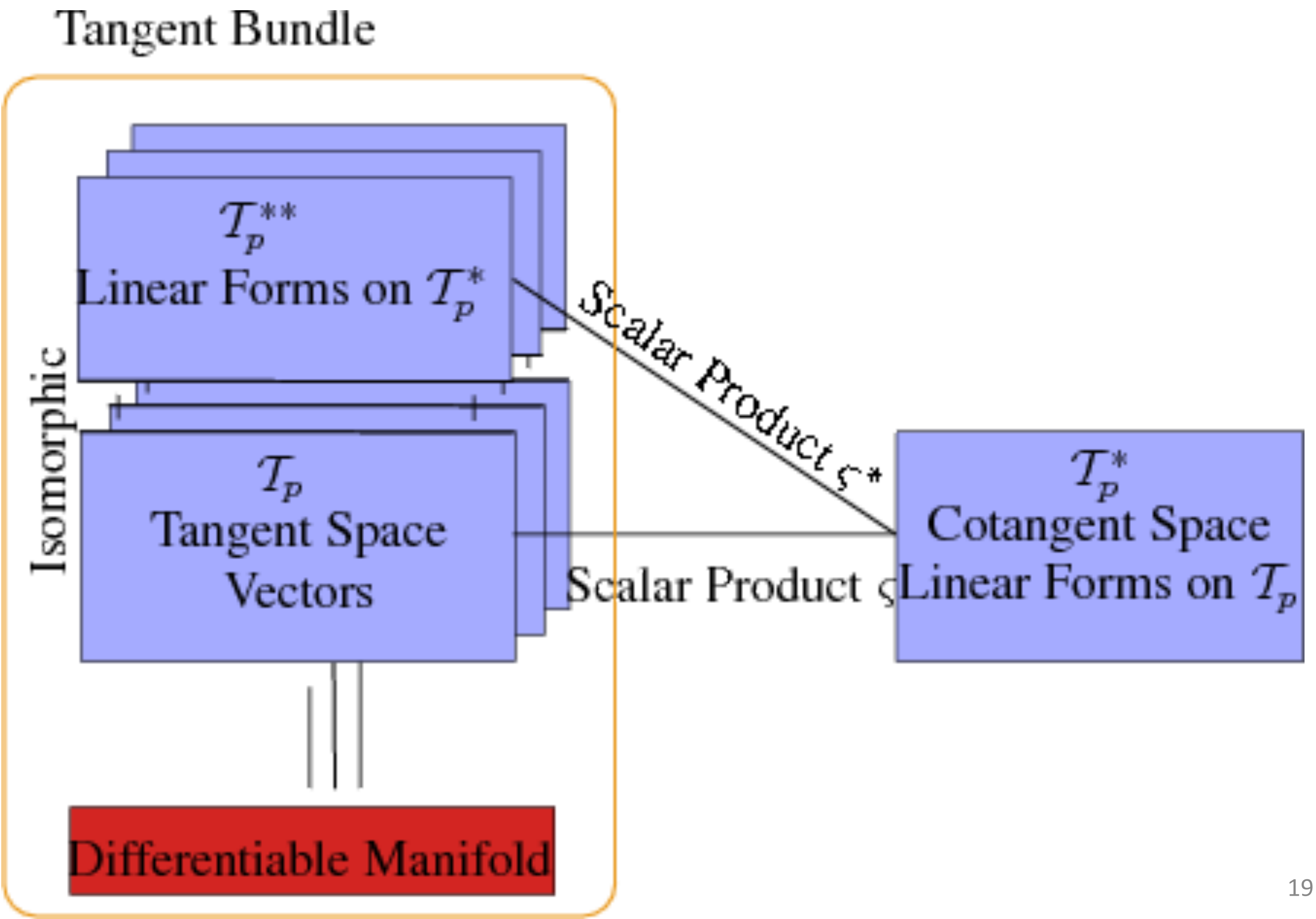
**Dual Pair if ($\omega$,$V$)= 1**

**Multivectors $V^{[\ldots]}$** **Forms $\omega_{[\ldots]}$**

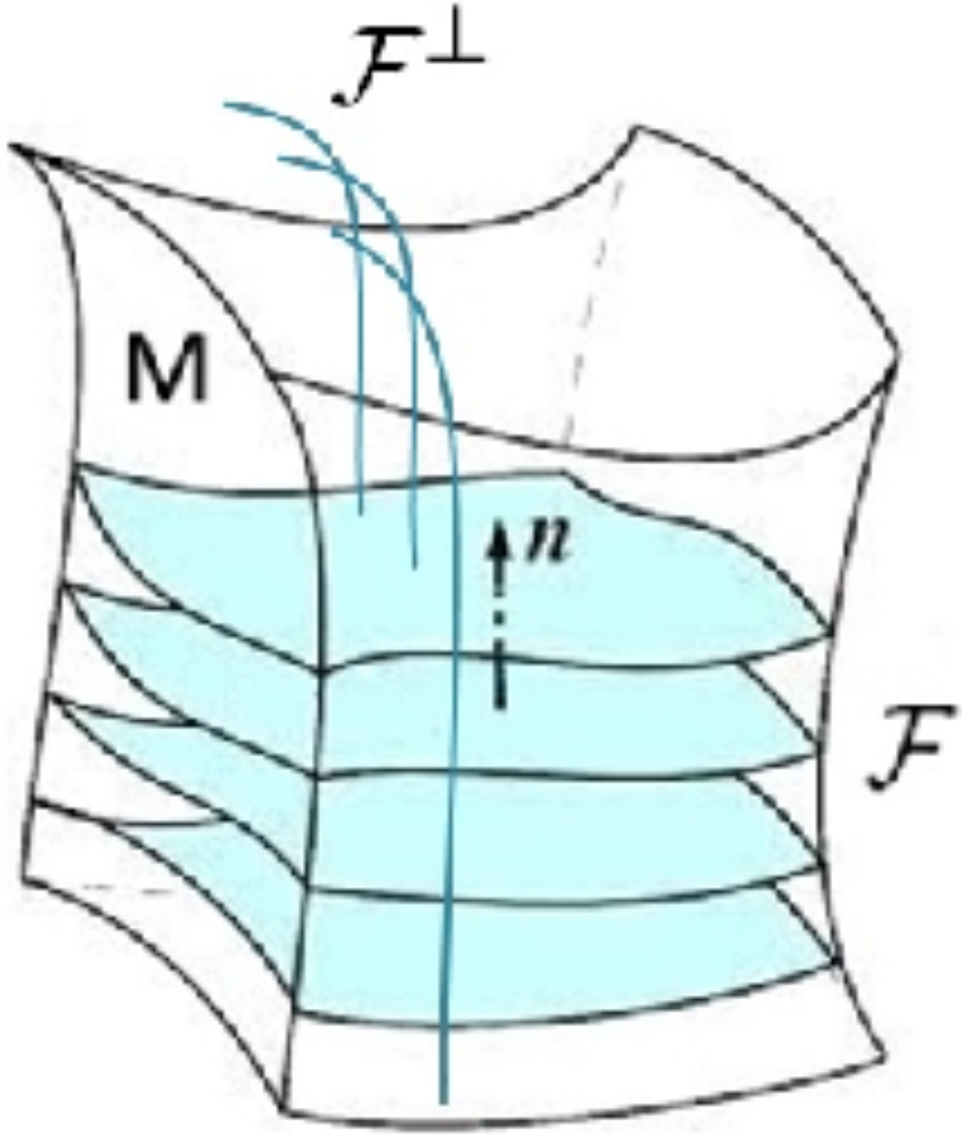
**$T(X_n)$ tangent space** **$T^*(X_n)$ cotangent space**

**Fibration** **Foliation**

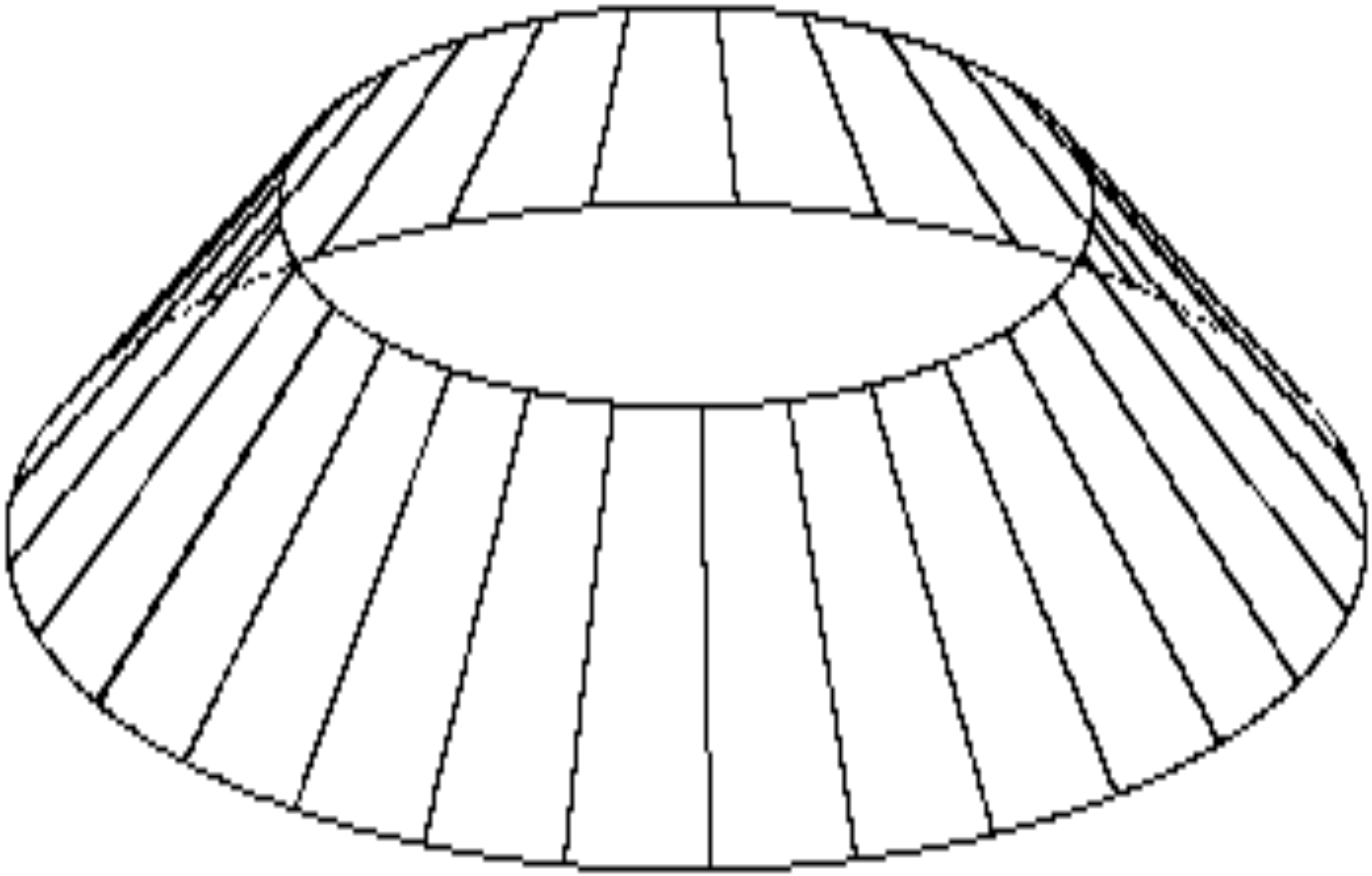
# Dual Spaces:
# Tangent $T\,(X_n)$, Cotangent $T^*(X_n)$



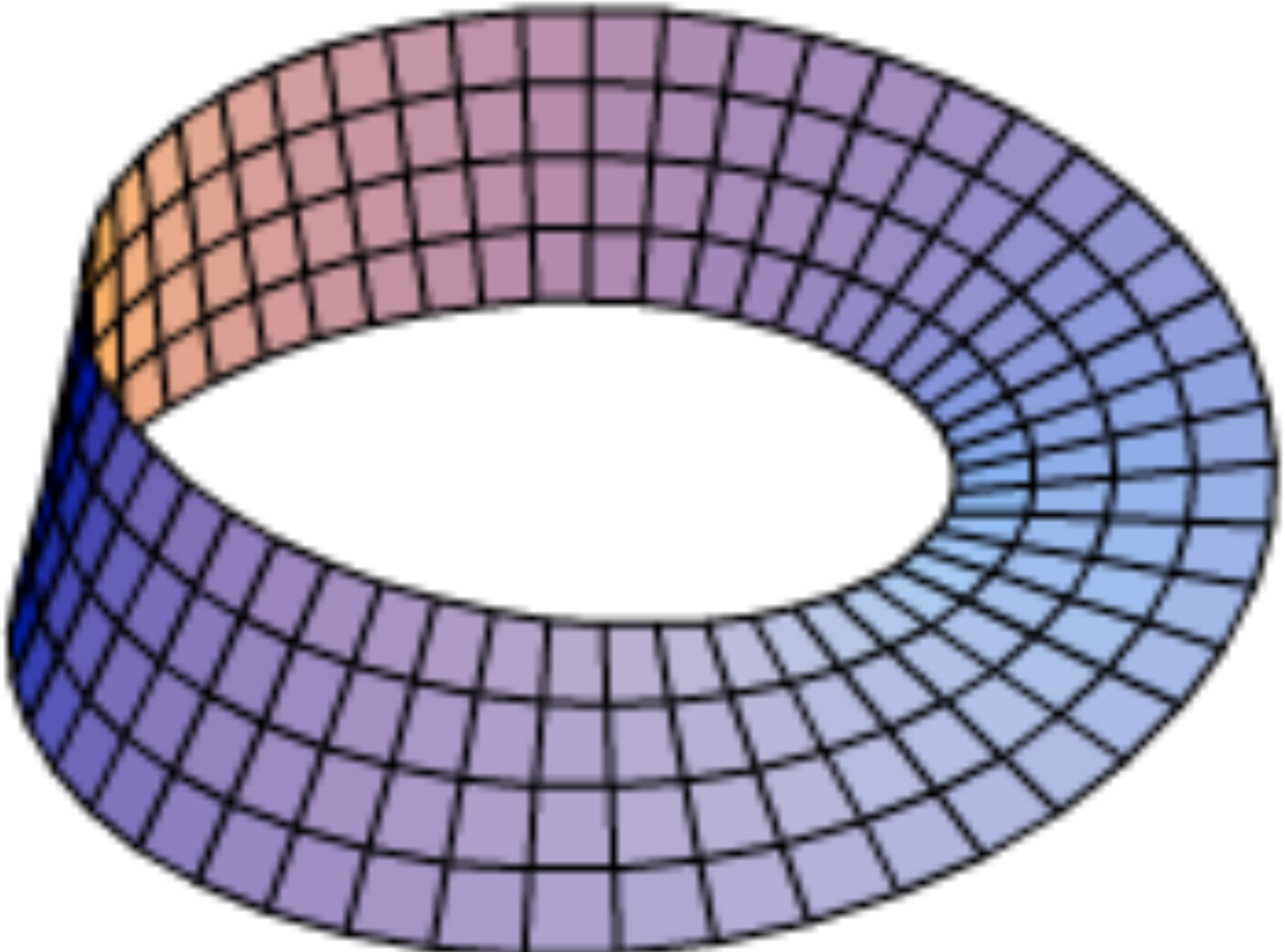
# Foliation



# Fibration

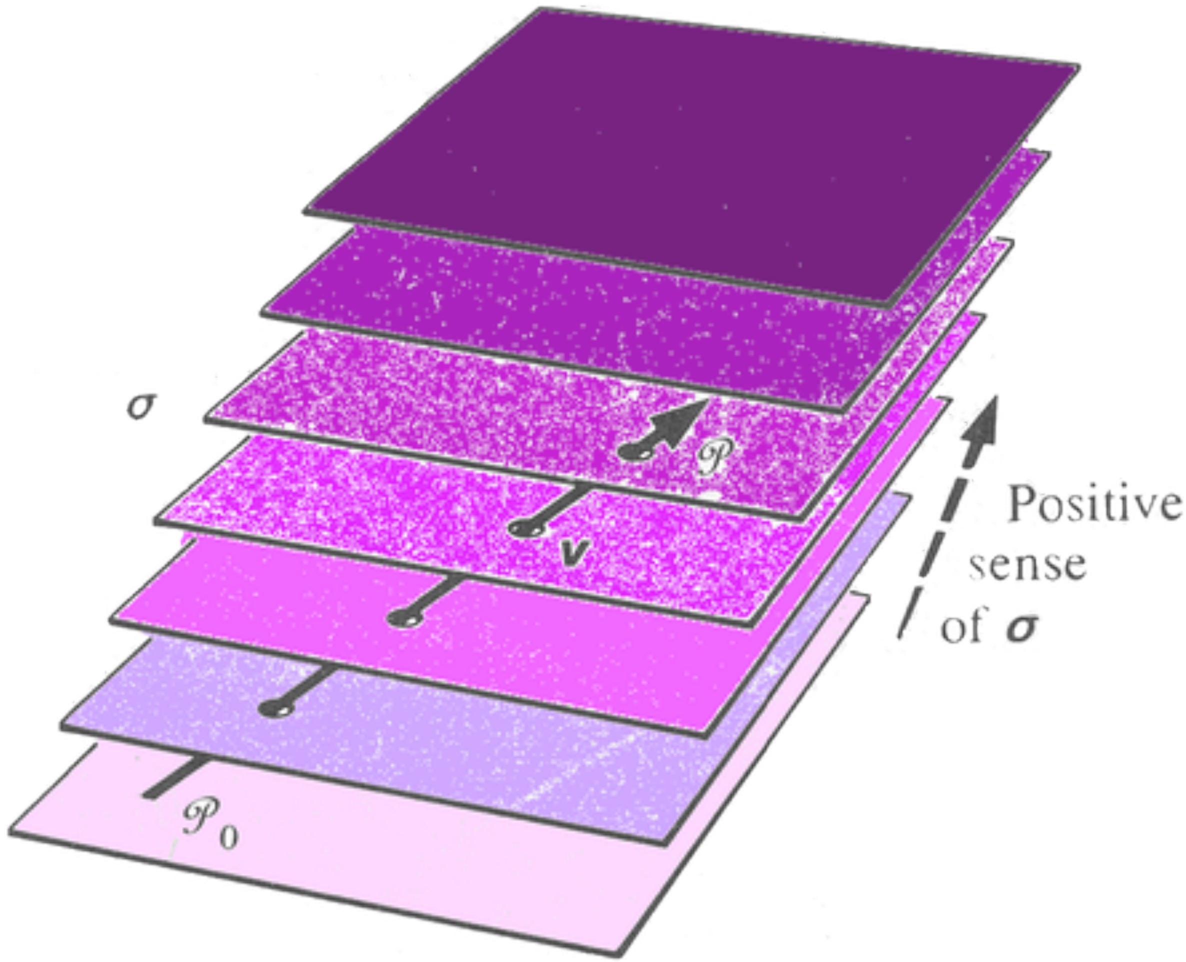


# Fibration and Foliation



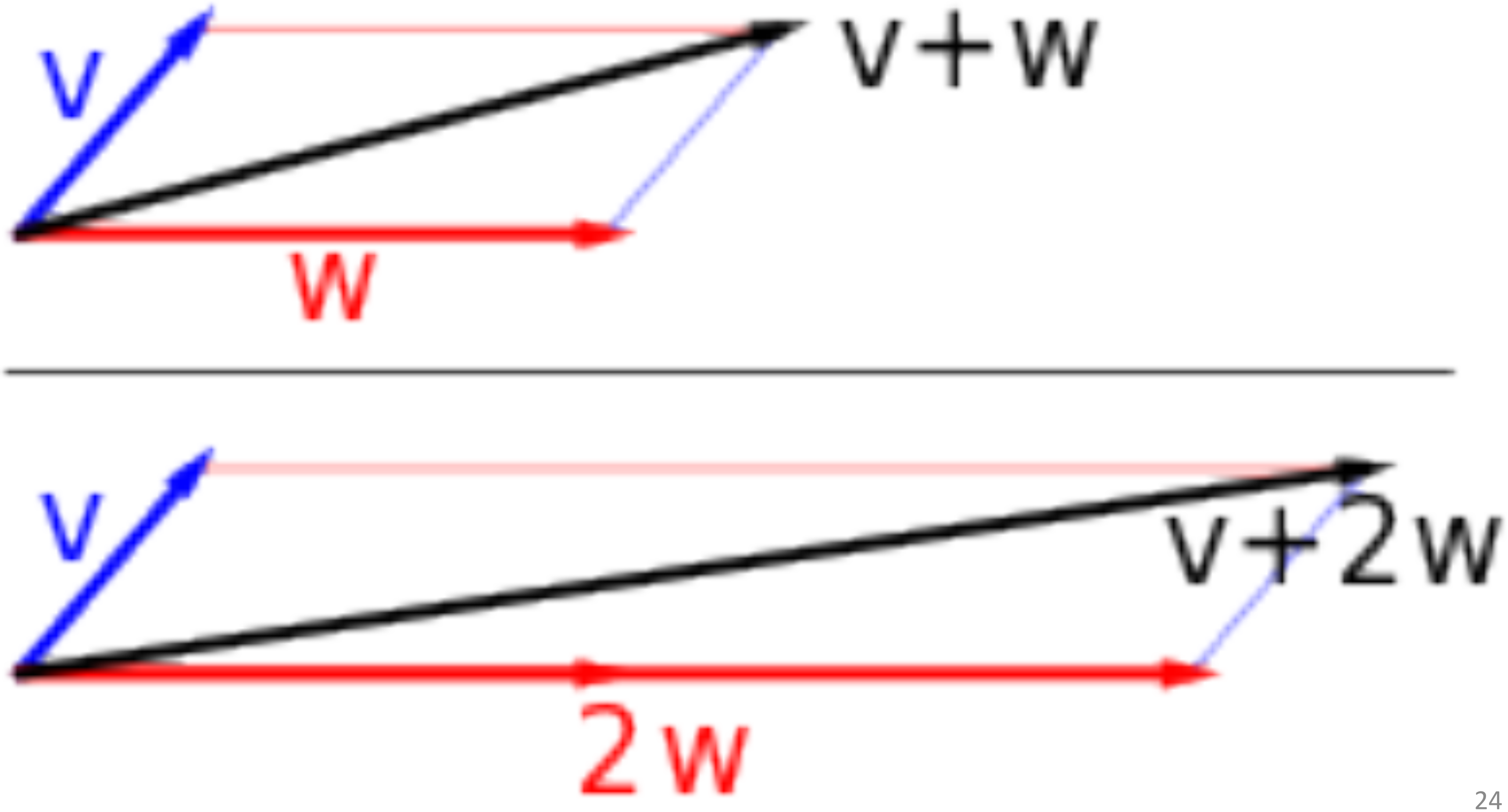
# Foliation and Dual Fibration

$(\omega, V) = 1$ , $V = d\mathbf{x}/d\sigma$



# Invariantce of Duality Under Certain Changes of *V*:

($\omega$,$V$+c$W$) = 1



# Some Things Einstein Got *Right!!*

1) The Hole Argument: Points of Space-Time have no inherent physical properties-- They inherit *all* of these properties from the Space-Time Structures, including all fields

Conclusion: No first order space-time structures or fields, no space-time!

If the water drains out of *this* bathtub, it takes the bathtub with it!

# GR is a Background-Independent Theory

**The contrast between general relativity and all previous theories:**

**Background-dependent theory:** **Fixed and given space-time stage, on which the drama of physics unfolds.**

**Background-independent theory**: **No actors, no stage, no anything.**

**Einstein put it this way:**

**"Space-time does not claim existence on its own, but only as a structural quality of the field."**

# “Relativity and the Problem of Space“ (1952)

**On the basis of the general theory of relativity ... space as opposed to ‘what fills space’ ... has no separate existence. If we imagine the gravitational field to be removed, there does not remain a space of the type [of SR], but absolutely nothing, not even a ‘topological space’.**

# Realizing Einstein's Vision

**The concepts of fiber bundles and sheaves enable a mathematical formulation of general relativity consistent with Einstein's vision.**

**But we have no time to go into this. See, e.g., the PowerPoint**

**Structures and Categories**
**John Stachel**
**Center for Einstein Studies**
**Boston University**
**Florence Category Day**
**16 June 2010**

**And the paper:**

"The Hole Argument and Some Physical and Philosophical Implications,"
*Living Reviews in Relativity*, 2014

# Differentiable Manifold: First Order Structures

**Pseudo-Metric** with Lorentz signature (+ ---), Minkowski Space is the tangent space with **symmetry group O(3,1)**

**Affine Linear Connection**, covariant derivative and affine curvature tensor with symmetry group **Aff(4)**

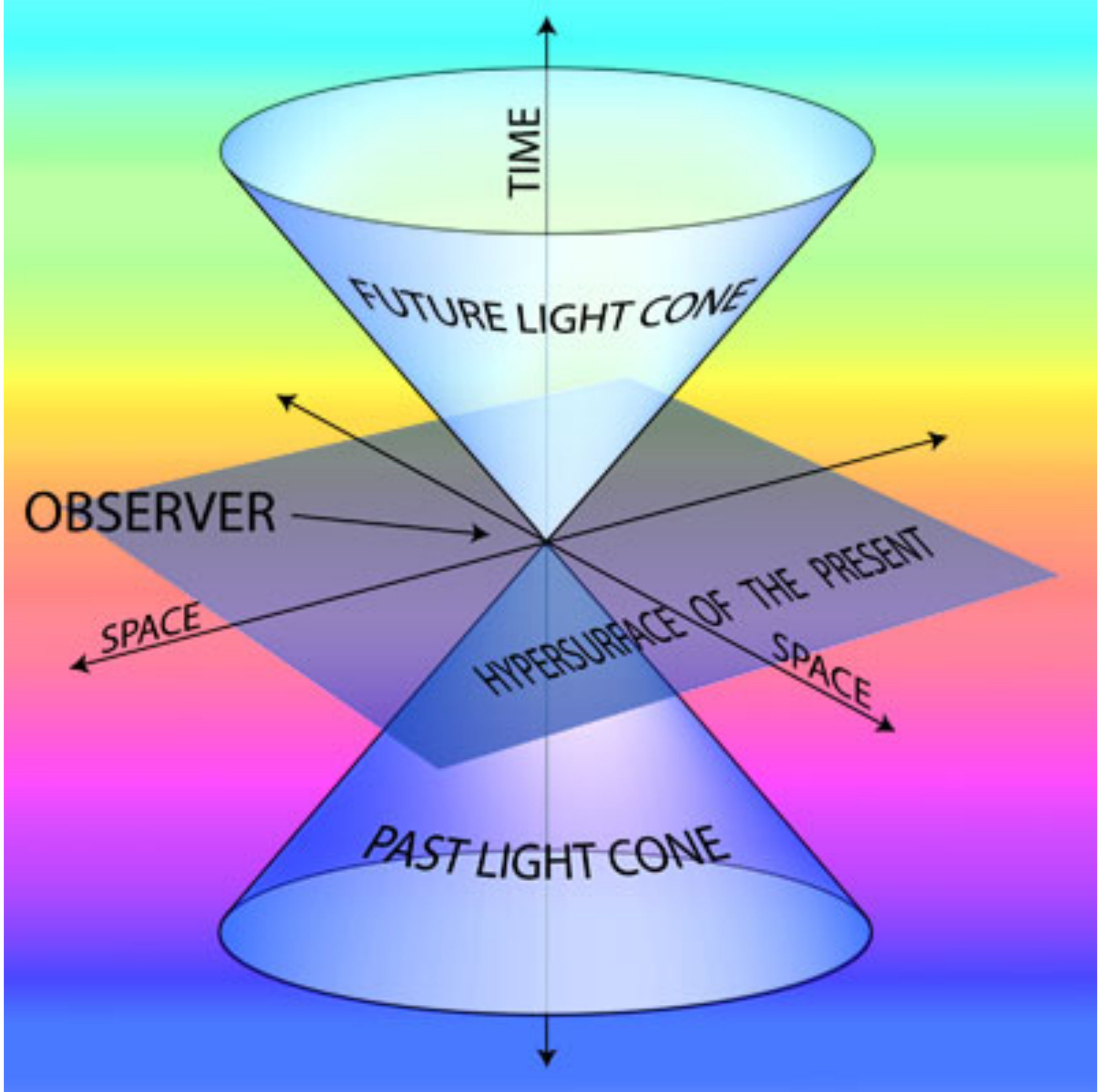
**Compatibility Conditions**

# Reasons for Adopting the Equivoluminar Condition $\Delta = \pm 1$

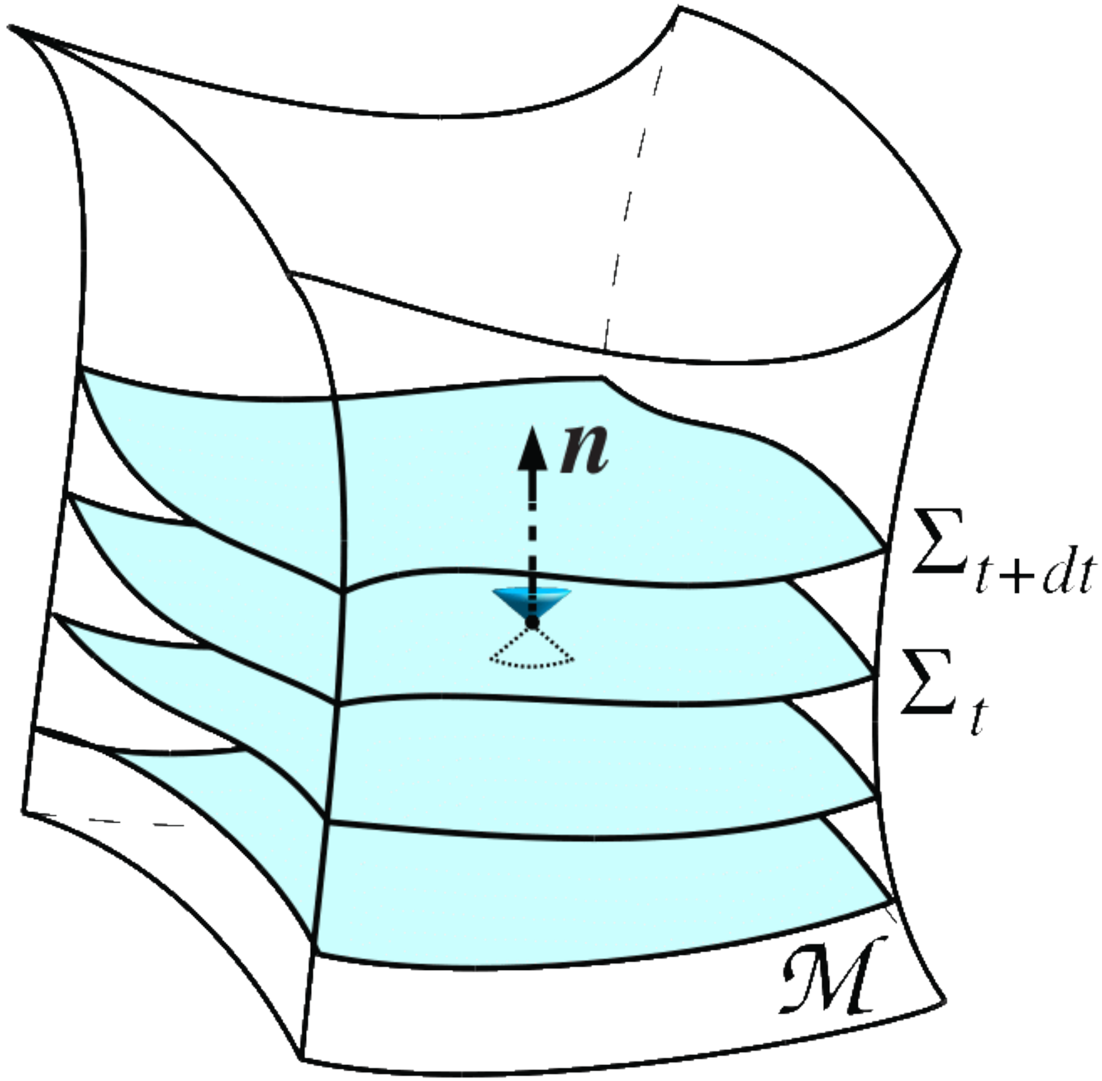
**1) To restrict the pseudo-orthogonal group locally and the pseudo-orthogonal metrics globally, resulting in the invariant decomposition of the pseudo-orthogonal metric into an equivoluminar (det = -1) pseudo-orthogonal metric and a scalar field.**

**2) To restrict the affine geometry locally and the linear affine connection globally, resulting in the invariant decomposition of the connection into a trace free connection and a one form.**

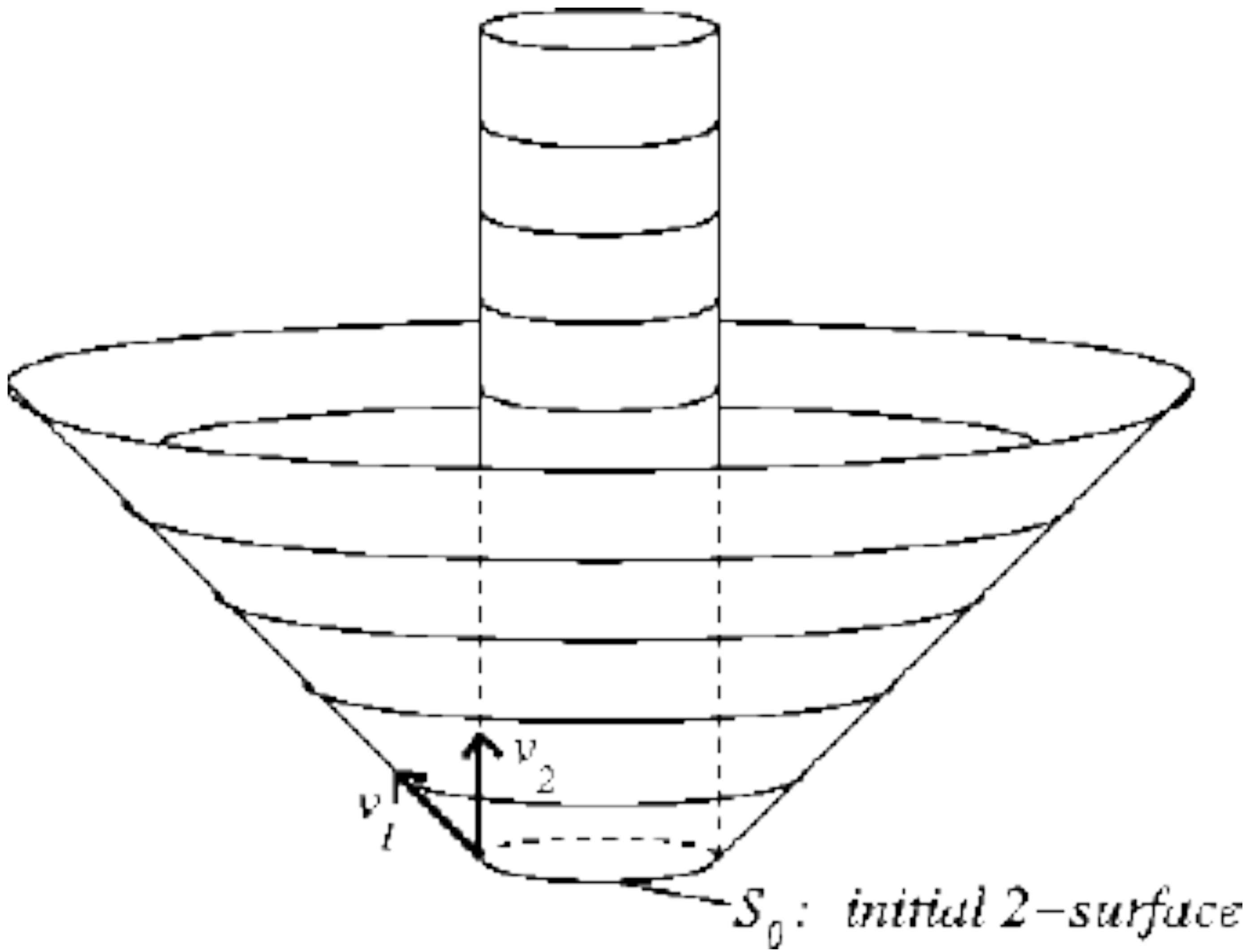
# SR: Minkowski Space-Time



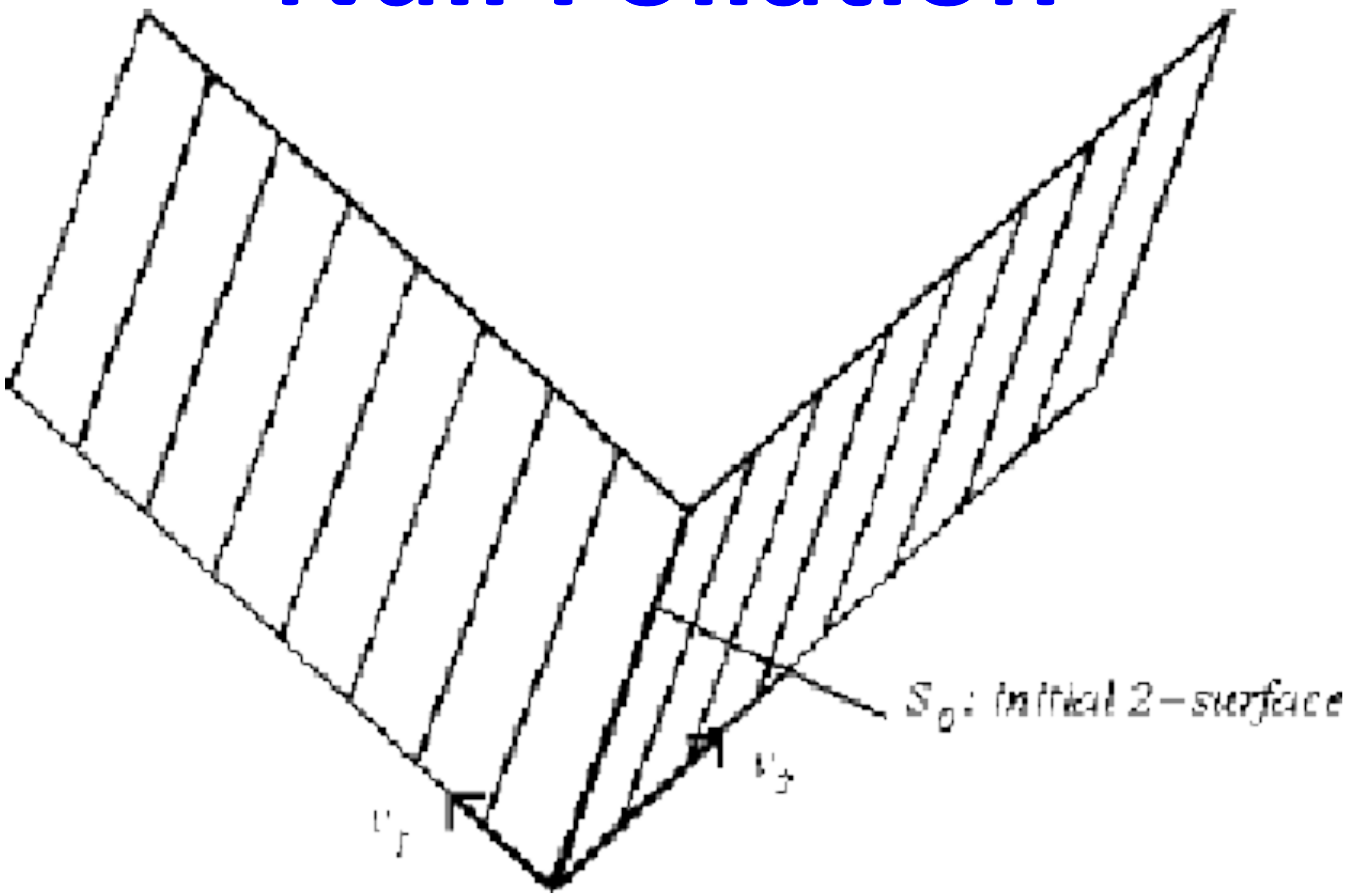
# Lorentz Metric: Spacelike Foliation, Timelike Normal



# Lorentz Metric: Null Hypersurface



# Lorentz Metric: Double Null Foliation



# Conformal Two-Structure

**An analysis of the field equations shows that the two gravitational degrees of freedom may be chosen, with no constraints, as the conformal 2- structure.**

**The dynamical equations then consist of two equations, which propagate the conformal 2-structure.**

**See R.A. d'Inverno and J. Stachel, *J. Math, Phys.* 19 (1978): 2447.**

# First Order Dualities

## Pseudo-Metric

$g_b(V,V)$ $g^{\#}(\omega, \omega)$

| | |
|---|---|
| $V$ is timelike if $g_b(V,V) > 0$ | $\omega$ is spacelike if $g^{\#}(\omega, \omega) > 0$ |
| " " null " " $= 0$ | " " null " " $= 0$ |
| " " spacelike " " $< 0$ | " " timelike " " $< 0$ |

## Metric Connection

$D_b$ $D^{\#}$

$\{^{m}{}_{ab}\}_b$ $\{^{nm}{}_{a}\}^{\#}$

# Reminder: "Flat" versus "Sharp"

Quantities formed from the covariant metric $g_{\mu\nu}$ will be called "flat" and distinguished by the symbol "$_b$"; while quantities formed from the contravariant metric $g^{\mu\nu}$ will be called "sharp" and distinguished by the symbol "#".

# “Flat” Christoffel Symbols

The “flat” Christoffel symbols of the first kind, are defined by:

$$[\lambda\nu, \mu]_b = ½(-\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu}).$$

The usual “flat"metric connection $\{^\kappa{}_{\lambda\nu}\}$ is defined as:

$$\{^\kappa{}_{\lambda\nu}\} = [\lambda\nu,\mu]_b \, g^{\kappa\mu} .$$

It follows that:

$$D_\mu g_{\lambda\nu} = \partial_\mu g_{\lambda\nu} – [\mu\lambda,\nu]_b - [\mu\nu,\lambda]_b$$

# "Sharp" Christoffel Symbols

The "sharp" Christoffel symbols of the first kind, are defined by:

$$[\lambda\nu, \mu]^{\#} = ½(g^{\kappa\mu}\partial_k g^{\lambda\nu} - g^{\kappa\nu}\partial_k g^{\mu\lambda} - g^{\kappa\lambda}\partial_k g^{\nu\mu}).$$

The usual "flat"metric connection $\{^{\mu}{}_{\alpha\beta}\}$ can now be defined as:

$$\{^{\mu}{}_{\alpha\beta}\} = [\lambda\nu, \mu]^{\#} g_{\nu\alpha}\, g_{\lambda\beta} \;.$$

# “Sharp” Christoffel Symbols

But we now define “sharp” Christoffel symbols of the second kind:

$$\{^{\nu\mu}{}_{\alpha}\}^{\#} =_{\mathrm{df}} [\lambda\nu, \mu]^{\#} g_{\lambda\alpha}\,,$$

from which it easily follows that:

$$D_{\alpha}\, g^{\lambda\nu} = \partial_{\alpha}\, g^{ln} + \{^{\nu\lambda}{}_{\alpha}\}^{\#} + \{^{\lambda\nu}{}_{\alpha}\}^{\#} = 0.$$

We shall see that there is no need for the “flat” Christoffel symbols.

# "Sharp" Covariant Derivatives

It follows from this definition that:

$$D^{\mu}g^{\lambda\nu} = g^{\kappa\mu}\partial_{k}g^{\lambda\nu} + [\mu\nu,\lambda]^{\#} + [\mu\lambda,\nu]^{\#}.$$

Similarly, by using the sharp Christoffel symbols $\{^{\nu\mu}{}_{\alpha}\}^{\#}$, we can show that

$$D^{\mu}V^{\nu} = g^{\kappa\mu}\partial_{\kappa}V^{\nu} + \{^{\mu\nu}{}_{\alpha}\}^{\#}V^{\alpha}$$

and

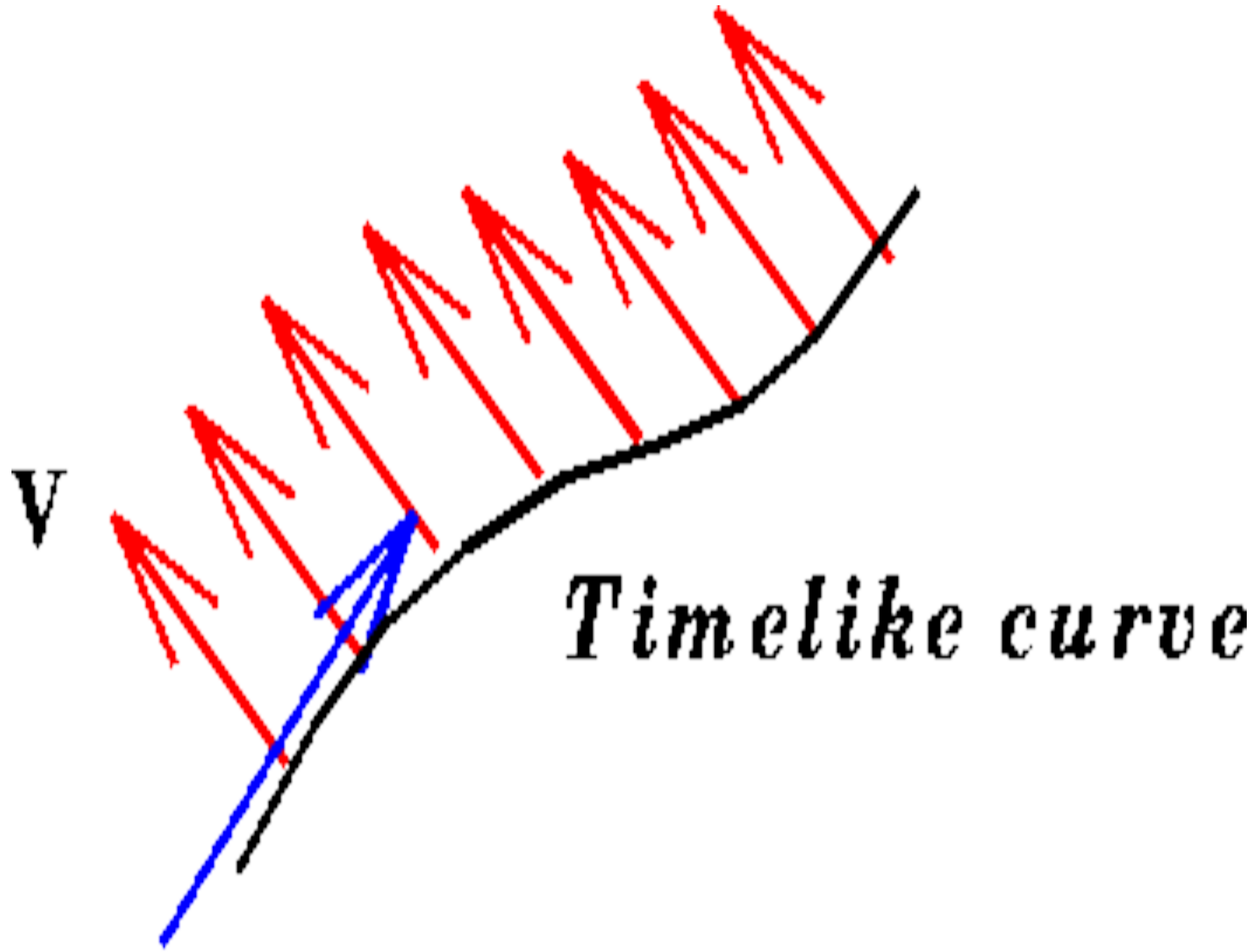
$$D^{\mu}\omega_{\nu} = g^{\kappa\mu}\partial_{\kappa}\omega_{\nu} - \{^{\mu\lambda}{}_{\nu}\}^{\#}\,\omega_{\lambda}\,.$$

# The Inertial Structure

This determines the motion of freely-falling (i.e., net force-free) bodies ("particles") in both classical and special-relativistic physics:

They follow the straightest time-like, inertial paths (autoparallels) of space-time-- straight lines for the "flat" (curvature = 0) space-times of both N-G and Minkowski space-time.

# Parallel Transport Along a Time-Like Curve



# Autoparallel Curves

**A curve is a parametrized path.**

**If the tangent vector field to a curve is parallel transported into itself along the curve, it is called an autoparallel curve (& often loosely called "a geodesic")**

# Affine Connection

**The affine connection $\Gamma^{\kappa}{}_{\mu\nu}$ represents the inertio-gravitational structure of space-time.**

**The affine connection $\Gamma^{\kappa}{}_{\mu\nu}$ determines the autoparallel behavior of free particles**

**We assume the connection is symmetric:**

**The torsion tensor $S^{\kappa}{}_{[\mu\nu]} = \Gamma^{\kappa}{}_{[\mu\nu]} = 0$ .**

# Law of Inertia and Autoparallels

**The affine connection $\Gamma^{\kappa}{}_{\mu\nu}$ determining the autoparallel behavior of free particles can**

**be split into:**

**The trace-free projective parameters $\Pi^{\kappa}{}_{\mu\nu}$, which determine the autoparallel paths.**

**The trace of the affine connection $\Theta_{\mu} = \Gamma^{\nu}{}_{\mu\nu}$, which determines a parameterization, turning the paths into autoparallel curves**

# Non-Flat Affine Connection

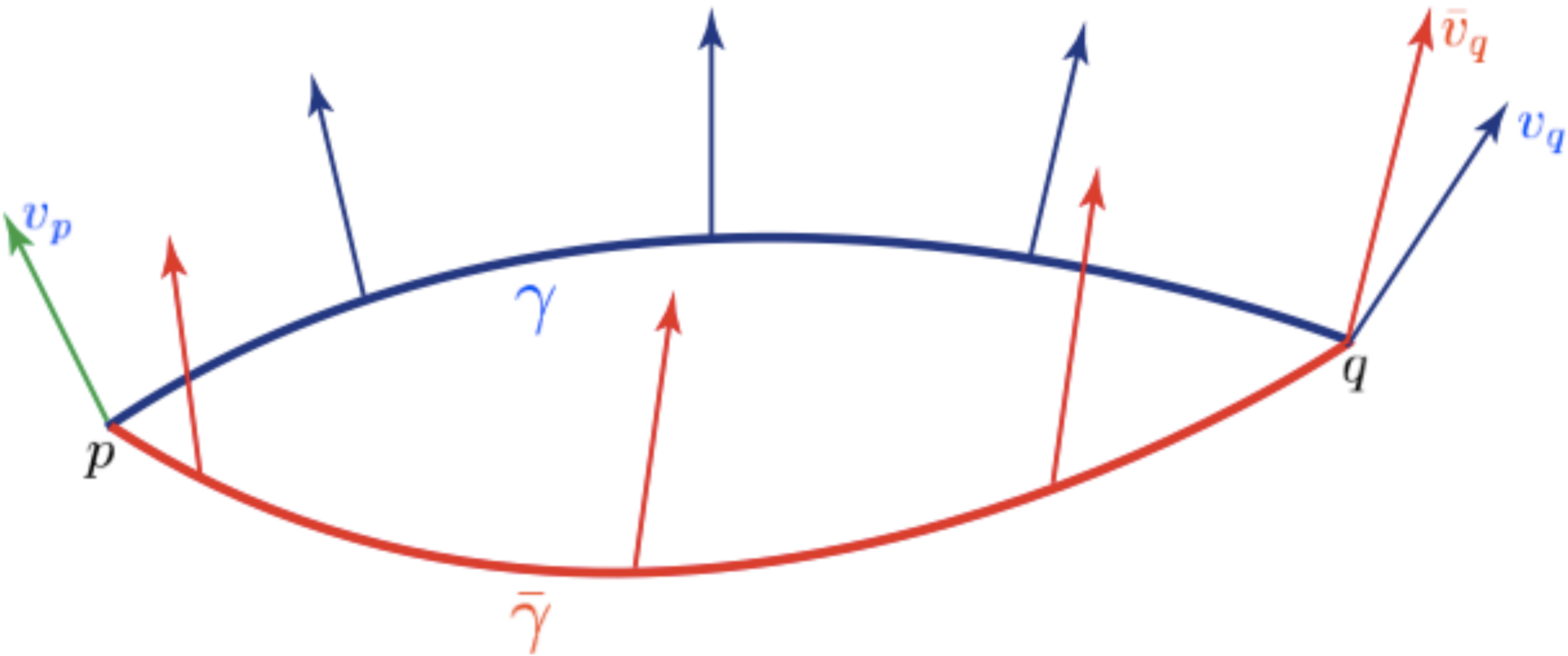
**The affine connection $\Gamma^{\kappa}{}_{\mu\nu}$ represents the inertio-gravitational structure of space-time.**

**The affine connection $\Gamma^{\kappa}{}_{\mu\nu}$ determines the autoparallel behavior of free particles**

**We assume the connection is symmetric:**

**The torsion tensor $S^{\kappa}{}_{[\mu\nu]} = \Gamma^{\kappa}{}_{[\mu\nu]} = 0$ .**

# Affine Curvature



# Affine Curvature Tensor

The affine curvature tensor is

$$A_{\nu\mu\lambda}{}^{\kappa} \equiv 2\partial_{[\nu}\Gamma^{\kappa}{}_{\mu]\lambda} + 2\Gamma^{\kappa}{}_{[\nu|\rho|}\Gamma^{\rho}{}_{\mu]\lambda}$$

The affine Ricci tensor

$$A_{\mu\lambda} \equiv A_{\kappa\mu\lambda}{}^{\kappa} ,$$

The homothetic curvature tensor

$$V_{[\nu\mu]} \equiv A_{\nu\mu\kappa}{}^{\kappa} .$$

$A_{\mu\lambda}$ is not necessarily symmetric, indeed:

$$A_{[\mu\lambda]} = -1/2V_{[\mu\lambda]} .$$

# Curvature and the Inertio-Gravitational Field

Does the vanishing of the curvature tensor imply the vanishing of the inertio-gravitational field?

Many current authors say "Yes"

Einstein said a resounding:

"No!!"

# Einstein to von Laue 1950

"It is true that in that case [the components of the curvature tensor] vanish, so one might say: "There is no gravitational field present." However, what characterizes the existence of a gravitational field from the empirical standpoint is the non-vanishing of the $\Gamma^{l}_{ik}$ …

# Einstein to von Laue 1950

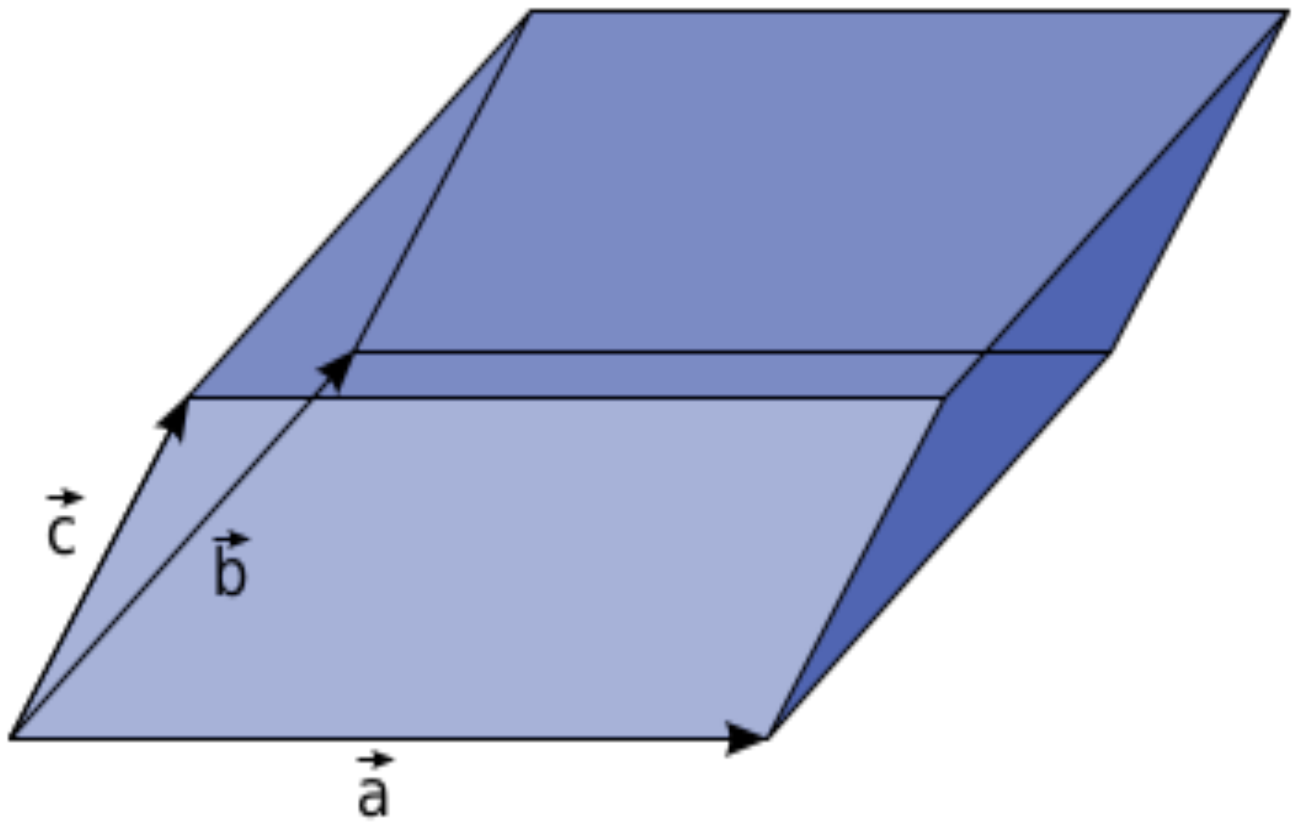
**If one does not think in such intuitive (*anschaulich*) ways, one cannot comprehend (be*greifen*) why something like curvature should have anything to do with gravitation in the first place. In that case, no reasonable person would have hit upon anything. The key to understanding the equality of inertial and gravitational mass would have been missing.”**

# The Volume Structure $\mathcal{V}$

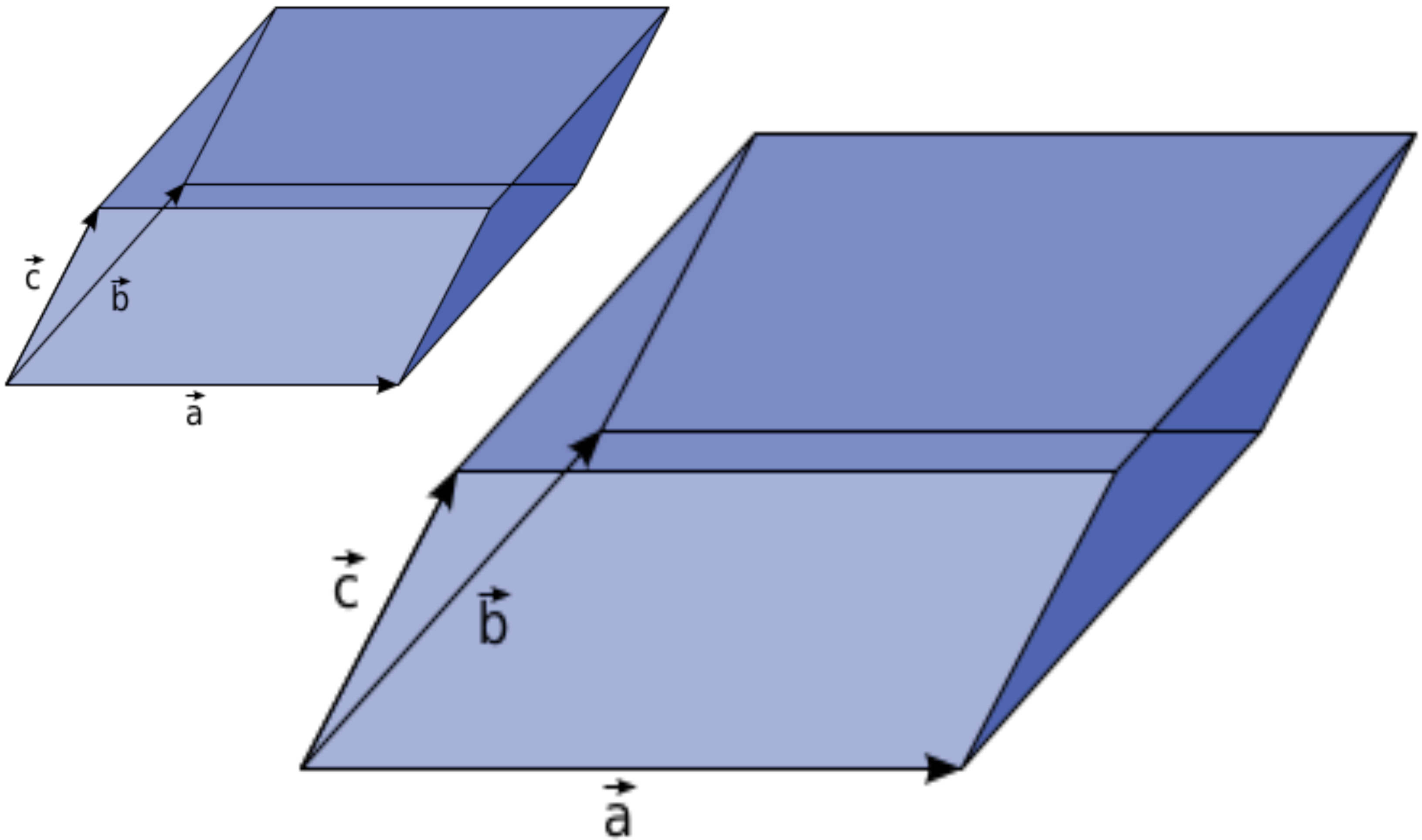
**Picks out an equivalence class of sets of basis vectors defining a unit four-volume in space-time.**

**Two sets are equivalent if they are related by an equiaffine transformation ε *SL*(4, *R*)**

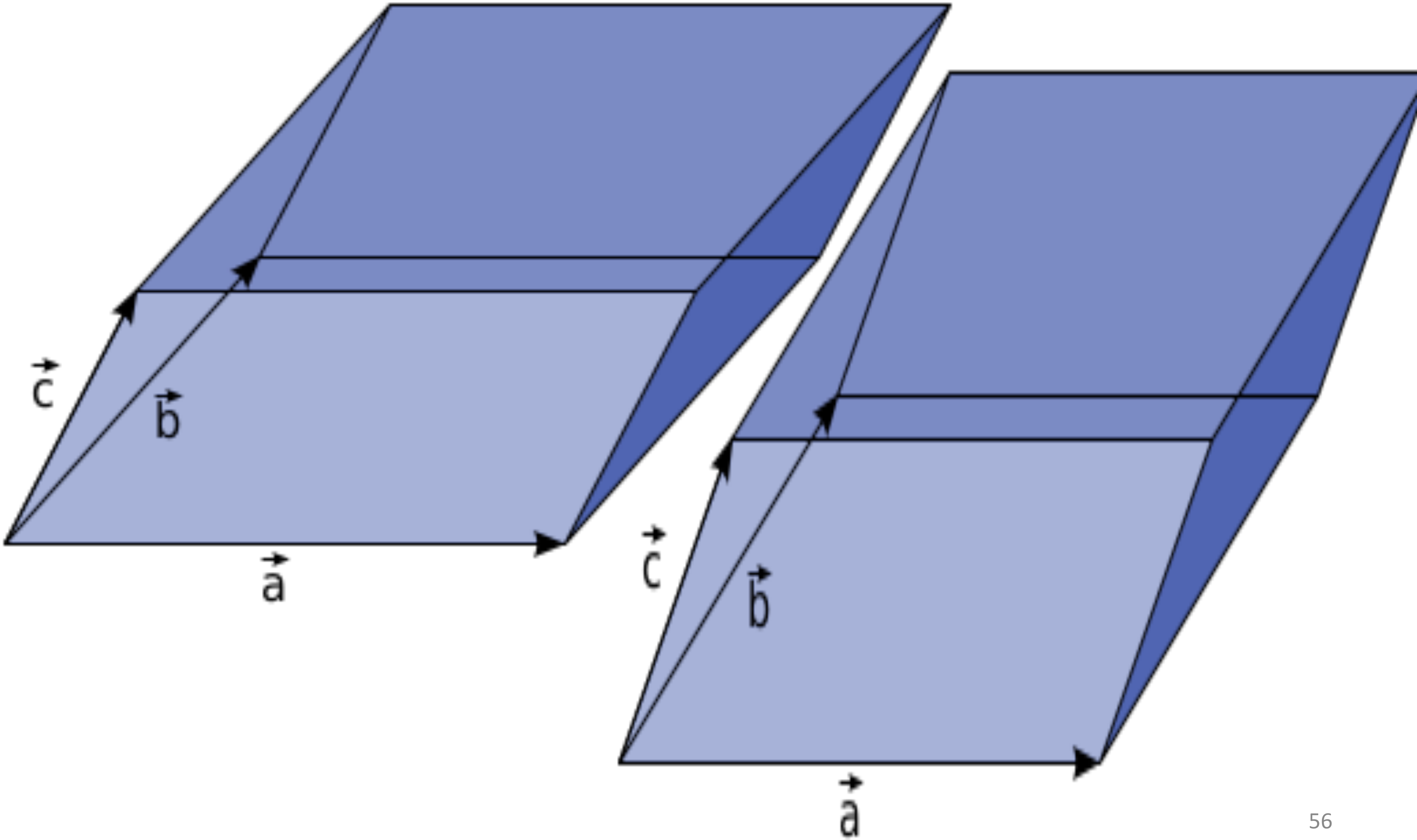
# The Volume Structure $V$



# Similarity Transformation



# Equiaffine Transformation



# Compatibility Conditions: Equi-Affine Spaces

**If the affine connection and the first prolongation of the volume structure are compatible, then parallel transport of volumes is path-independent, and the connection is called equiaffine, with symmetry group *SAff*(4).**

# Compatibility Conditions: Metric and Connection

GR includes compatibility conditions between the chrono-geometry and the inertio-gravitational field.

The usual mathematical form:

The covariant derivative of the covariant metric with respect to the connection vanishes:

$$D_{\kappa} g_{\mu\nu} = 0.$$

# Compatibility Condition

$$D_\kappa g_{\mu\nu} = 0$$

**The usual physical interpretation:**

**1) proper spatial and temporal intervals are preferred parameters along affine autoparallel (now also geodesic) curves;**

**2) rods and clocks still read proper spatial and temporal intervals as they move in the inertio-gravitational field.**

# Form of Compatibility Condition From the Lagrangian

**Let $\mathfrak{g}^{\mu\nu} = (-g^{\#})^{-1/2} g^{\mu\nu}$. The Lagrangian density is:**

$$\mathcal{L} = \mathfrak{g}^{\mu\nu} A_{\mu\nu}(\Gamma, \partial\Gamma).$$

**Variation of the torsion-free connection $\Gamma^{\nu}{}_{\mu\lambda}$:**

$$\delta A_{\mu\lambda} = D_{\nu}(\delta\Gamma^{\nu}{}_{\mu\lambda}), \text{ so}$$

$$\delta\mathcal{L} = \mathfrak{g}^{\mu\nu} \delta A_{\mu\nu}$$

$$= D_{\kappa}[\mathfrak{g}^{\mu\nu} \delta\Gamma^{\kappa}{}_{\mu\nu}] - D_{\kappa}(\mathfrak{g}^{\mu\nu})\delta\Gamma^{\kappa}{}_{\mu\nu}.$$

**The first term is a total divergence and can be eliminated. So the resulting equation is:**

# Compatibility Condition

**The form of the compatibility condition is**

$$D_\kappa g^{\mu\nu} = 0\,,$$

**a condition on the contravariant metric density:**

$$= \partial_\kappa g^{\mu\nu} + \Gamma^{\nu}{}_{\kappa\lambda} g^{\mu\lambda} + \Gamma^{\mu}{}_{\kappa\lambda} g^{\lambda\nu} - \Gamma_\kappa g^{\mu\nu}.$$

**Contracting with the inverse $g_{\mu\nu}$,**

$$\Gamma_\kappa = 1/2\, g_{\mu\nu} \partial_\kappa g^{\mu\nu} = \partial_\kappa \ln(-g^{\#})^{-1/2}$$

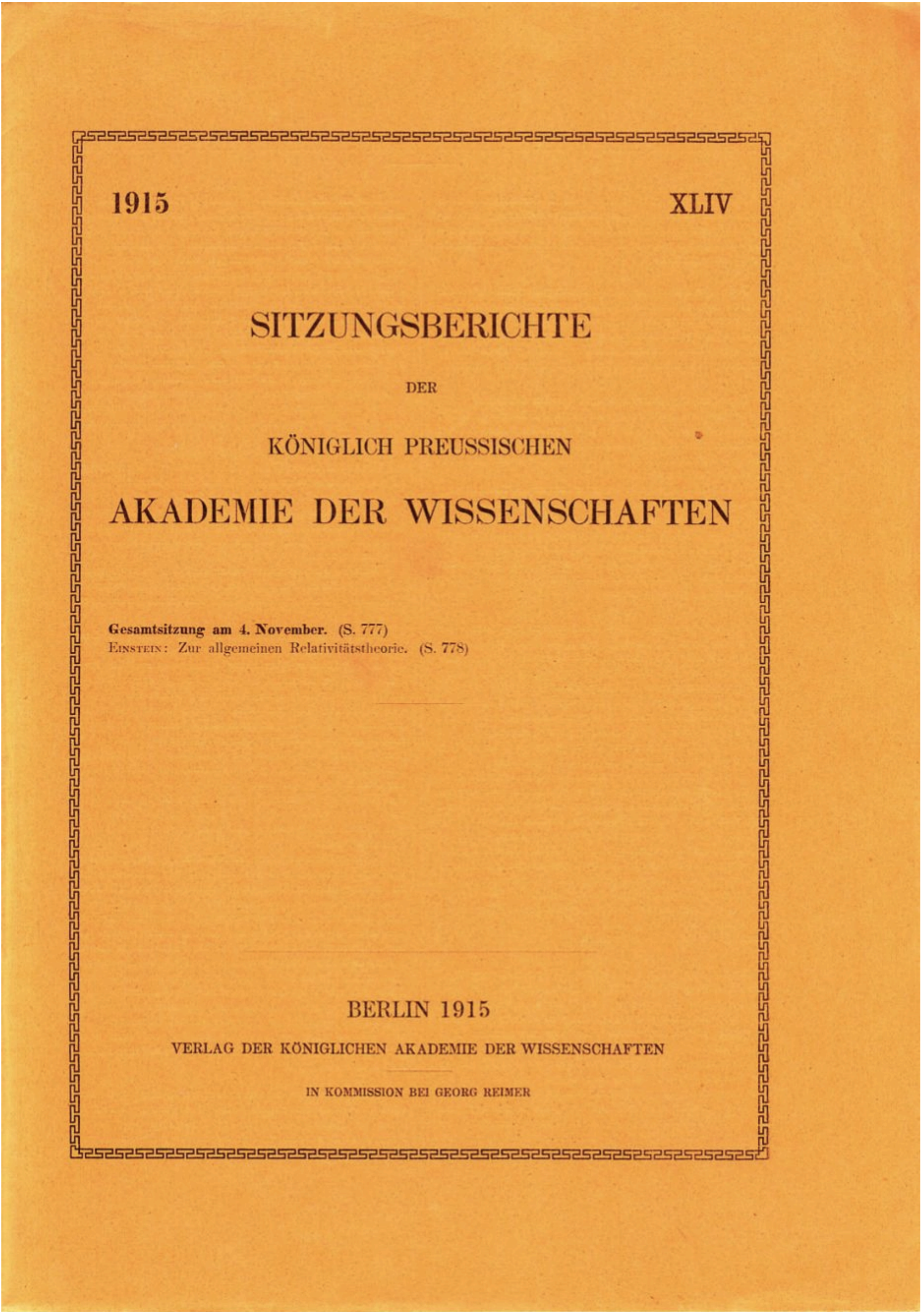
**which is the metric equiaffine condition.**

# Unimodular Transformations

**An equiaffine transformation in the tangent space $T_xM$ corresponds to a unimodular transformation in the space-time manifold *M*. The group *SL*(4, *R*) of equiaffine transformations corresponds to the group *SDIff* (*M*) of unimodular transformations, which also have determinant = 1.**

# Invariance Under *What* Group of Diffeomorphisms?

**"It is to be emphasized that we have no sort of justification for general covariance of the gravitational equations. … [W]e do not know if there is a general group of transformations, with respect to which the equations are covariant … The question of the existence of such a group … is the most important one …" (Einstein 1913)**



1915 XLIV

SITZUNGSBERICHTE

DER

KÖNIGLICH PREUSSISCHEN

AKADEMIE DER WISSENSCHAFTEN

**Gesamtsitzung am 4. November.** (S. 777)
Einstein: Zur allgemeinen Relativitätstheorie. (S. 778)

BERLIN 1915

VERLAG DER KÖNIGLICHEN AKADEMIE DER WISSENSCHAFTEN

IN KOMMISSION BEI GEORG REIMER

# Einstein had it right on 4 November 1915

# Einstein Adopts Volume-Preserving Transformations

**"Just as the special theory of relativity is based upon the postulate that all equations have to be covariant relative to linear orthogonal transformations, so the theory developed here rests upon the postulate of the *covariance of all systems of equations relative to transformations with the substitution determinant 1*."**

# *On General Relativity Theory*

"Because of the scalar character of $\sqrt{-g}$, one can simplify the basic formulas of the formation of covariants, as compared to those of general covariance; which in short means, the factors $\sqrt{-g}$ and $1/\sqrt{-g}$ no longer occur in the basic formulas, and the distinction between tensors and *V*-tensors drops out."

# Unimodular Transformations

**Today, we call ”*transformations with the substitution determinant* 1” unimodular transformations, and there are strong arguments for their adoption on the basis of numerous theories, both mathematical and physical, including all theories of gravitation based on the equivalence principle.**

# The Big Question

Why did Einstein move back from unimodular trans-formations to arbitrary diffeomorphisms?

No time to discuss this question here. See papers by JS and Kaća Bradonjić

# UCPR Lagrangian

The Lagrangian density is now:

$$\mathcal{L} = e^{\varphi}\, {}^{c}g^{\mu\nu} A_{\mu\nu}(\Pi^{\kappa}{}_{\mu\nu}, \Theta_{\mu}),$$

and variations are to be taken w.r.t.

$e^{\varphi}$ (1), ${}^{c}g^{\mu\nu}$ (9) field equations,

$\Pi^{\kappa}{}_{\mu\nu}$ (36) compatibilty conditions

$\Theta_{\mu}$ (4) equiaffine condition.

# Unimodular Group

**Under the unimodular group, $g^{\mu\nu}$ splits into a scalar field $e^{-\varphi}$ and a conformal metric ${}^{c}g^{\mu\nu}$ :**

$$g^{\mu\nu} = e^{-\varphi}\, {}^{c}g^{\mu\nu} .$$

**${}^{c}g^{\mu\nu}$ is a tensor with det = -1, and now represents the physical metric.**

**So: $-g^{\#} = e^{-4\varphi}$, $ln\,(-g^{\#})^{-1/2} = 2\varphi$ and**

$$\Gamma_{\kappa} = 2\partial_{\kappa}\varphi$$

**is the equiaffine condition.**

# Unimodular Group

The connection $\Gamma^{\mu}{}_{\nu\kappa}$ splits into a trace-free part, which is now the projective connection $\Pi^{\mu}{}_{\nu\kappa}$ and its trace, which is now a one form (covariant vector) $\Theta_{\kappa}$ .

The equiaffine condition is now

$$\Theta_{\kappa} = 2\partial_{\kappa}\varphi,$$

which is quite independent of the conformal and projective structures.

# Under Unimodular Transformations:

The conformal metric $^{c}g^{\mu\nu}$ determines Null hypersurfaces; together with the scalar field $e^{\varphi}$ it governs the:

propagation of massless fields.

The projective connection $\Pi^{\mu}{}_{\nu\kappa}$ determines Autoparallel paths; together with the one form $\Theta_{\kappa}$ it governs the:

behavior of massive bodies.

# Superpotentials

There are a denumerably infinite number of superpotentials, tensor densities differing in weight, from which the Einstein field equations may be deduced. Only one is conformally invariant, the one that defines a gravitational angular momentum complex such that the total angular momentum transforms as a free antisymmetric tensor.

# Gravitational Superpotential in UCPR

**In UCPR this superpotential:**

$$\mathcal{S}^{[\beta\kappa][\lambda\alpha]} = ({}^{c}g^{\beta\alpha}\,{}^{c}g^{\kappa\lambda} - {}^{c}g^{\beta\lambda}\,{}^{c}g^{\kappa\alpha}),$$

**which had been a tensor density,**

**is the one formed naturally from the *contravariant* conformal metric ${}^{c}g^{\mu\nu}$**

**and hence is a tensor– another argument for UCPR!**

# Some Things Einstein Got *Right!!* (but only later one)

**2) Space-Time Structures**

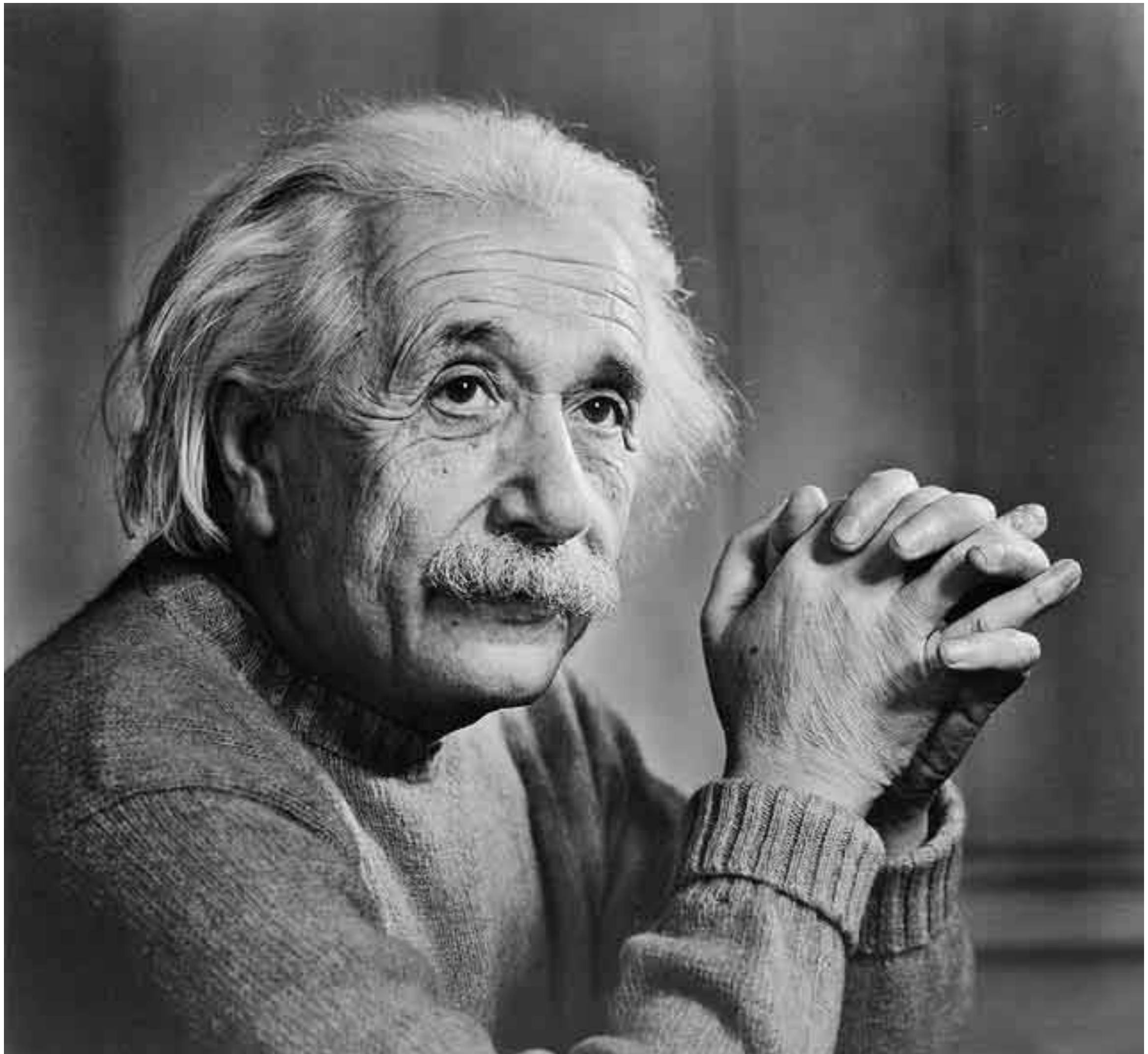
**a) chrono-geometry– first historically; it is represented mathematically by a metric tensor.**

**Equivalence principle: Inertia and Gravitation are "*wesensgleich*"– essentially the same.**

**b) inertio-gravitational field– first logically; it is represented mathematically by a linear connection.**

# Don’t take my word-

**Ask the maestro**



# Einstein, *The Meaning of Relativity,* Appendix to 5th ed (1955)

**The development … of the mathematical theories essential for the setting up of general relativity had the result that at first the Riemannian metric [chrono-geometry]was considered the fundamental concept on which the general theory of relativity and thus the avoidance of the inertial system were based.**

# Einstein, *The Meaning of Relativity*, Appendix to 5th ed (1955)

**Later, however, Levi-Civita rightly pointed out that the element of the theory that makes it possible to avoid the inertial system is rather the infinitesimal displacement field $\Gamma^{l}_{ik}$[the inertio-gravitational field]. The metric or the symmetric tensor field $g_{ik}$ which defines it is only indirectly connected with the avoidance of the inertial system in so far as it determines a displacement field.**

# Einstein's last publication, April 1955

[I]t seems to me that Levi-Civita's most important contribution lies in the following theoretical discovery: the most essential theoretical accomplish-ment of general relativity, namely the elimination of "rigid" space, i.e. of the inertial system, is only *indirectly* connected with the introduction of a Riemannian metric.

# Einstein's last publication, April 1955

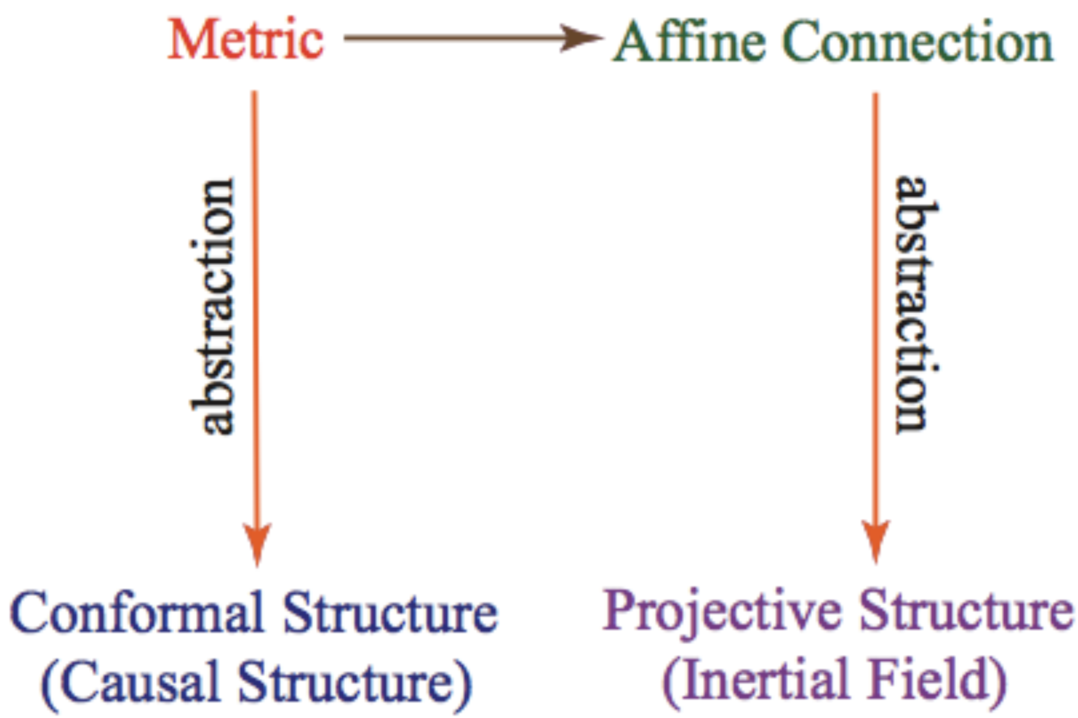
The immediately essential conceptual element is the "displacement field" ($\Gamma^l_{ik}$), which expresses the infinitesimal displacement of vectors. More explicitly, it replaces the parallelism of spatially separated vectors, posited with the help of an inertial system, by an infinitesimal operation. … In contrast to this, in a certain sense it is of secondary importance that a particular Γ-field may be derived from the existence of a Riemannian metric.

# Space-Time Structures

*Any* theory of gravitation– Newtonian, SR, GR– incorporating the equivalence principle must start from an inertio-gravitational field

These theories can only differ in the relation between this inertio-gravitational field and the chrono-geometry

# As We Have Seen, They Are Not Irreducible



# Curvature Tensors in UCPR

1) The **near field**, tied to the **sources** of the inertio-gravitational field, is best described by
the **projective curvature tensor**.

2) The **far field**, needed to **discuss gravitational radiation** that has escaped from its sources, is best described by
the **conformal curvature tensor**.

# Curvature Tensors in UCPR

**In contrast to the traditional ones, the conformal and projective curvature tensors defined in UCPR have non-vanishing Ricci tensors; so *they* can play a direct role in the field equations.**

# Tangent Space

**SR holds in the tangent plane, its metric $\eta$ is invariant under equiaffine transformations.**

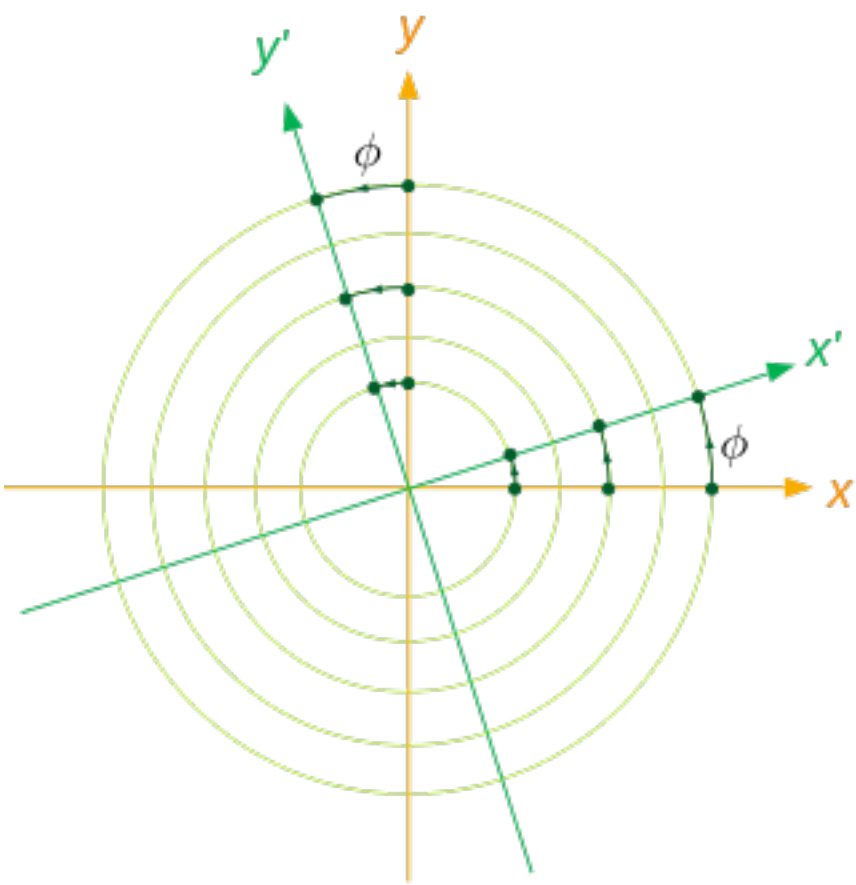
**Two basic types of tetrads:**

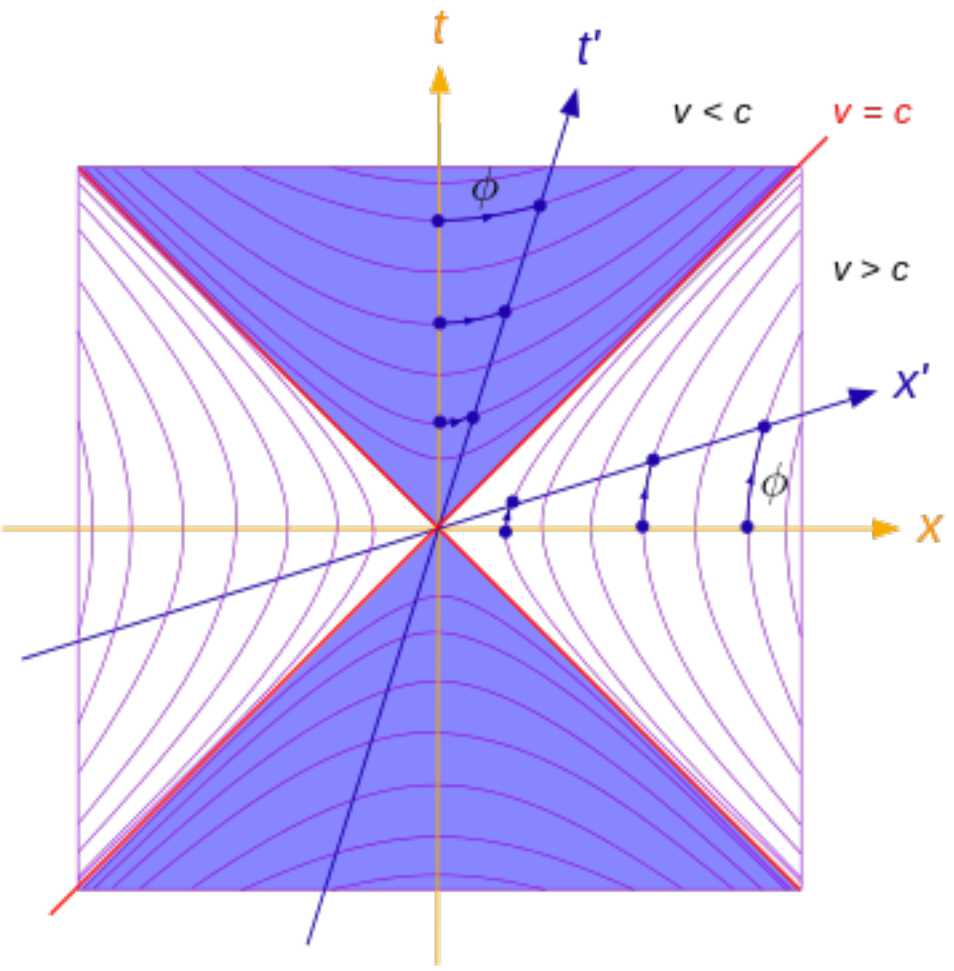
**On a time-like two-plane with units:**

**one time-like $e_t$, one space-like $e_s$**
**two null $k = 1/\sqrt{2}(e_t+e_s)$, $l = 1/\sqrt{2}(e_t-e_s)$**

**Plus two orthogonal space-like vectors on the orthogonal space-like two plane**

# Time-Like Two-Plane



# Conformal Structure

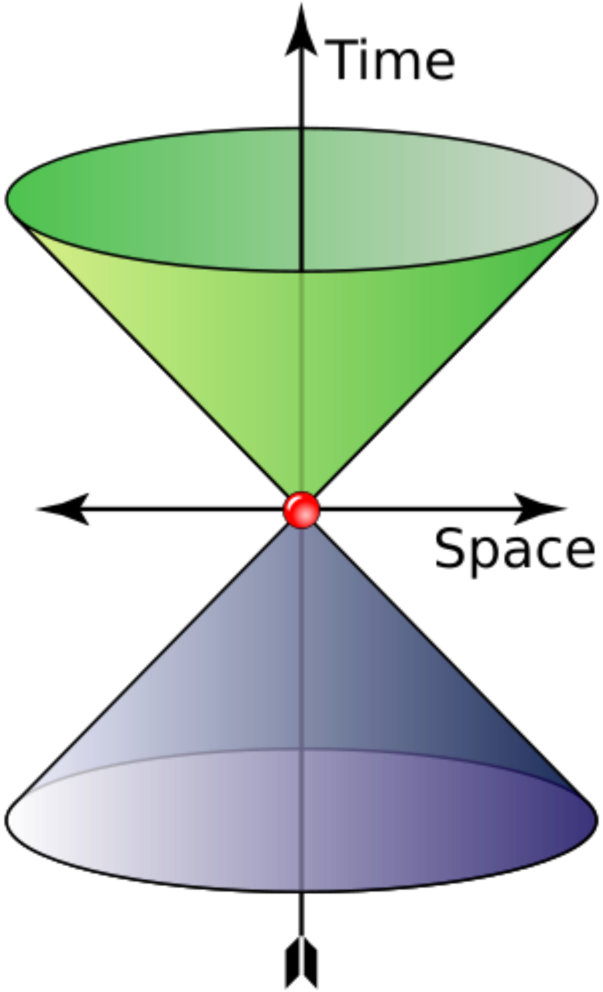
**Light cone**

**Angles**

**between vectors of same type**

**Ratios**

**of "lengths"**



# Conformal Structure

If one abstracts from the scale-changing property of the metric, one gets the **conformal structure** on the manifold.

Physically, this **conformal structure** represents the **causal structure of space-time**. The conformal structure of space-time determines the phases of **massless test fields**.

# Under Unimodular Transformations:

The (conformal) metric ${}^{c}g^{\mu\nu}$ determines
Null hypersurfaces,

Plus the scalar field $e^{\varphi}$ it governs the
propagation of massless fields

The (projective) connection $\Pi^{\mu}{}_{\nu\kappa}$ determines
Autoparallel paths,

Plus the one form $\Theta_{\kappa}$ it governs the
behavior of massive bodies.

# The Eikonal Equation for Light

$${}^{c}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} = 0\,,$$

**where $\Phi$ is the phase of the light wave, is invariant under conformal transformations**

**Light rays are the bicharacteristics of this equation**

# Characteristic Hypersurfaces

The **characteristic hypersurfaces** $\Phi = const$ are **null** since:

$$ {}^{c}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} = {}^{c}g^{\mu\nu}k_{\mu}k_{\nu} = 0, $$

in which $k_{\mu} = \partial_{\mu}\Phi$ is the **null covector** "**normal**" to the hypersurface. But

$$ {}^{c}k^{\mu} = {}^{c}g^{\mu\nu}k_{\nu} $$

actually lies **on** the null hypersurface:

$$ {}^{c}k^{\mu}\Phi_{,\mu} = 0. $$

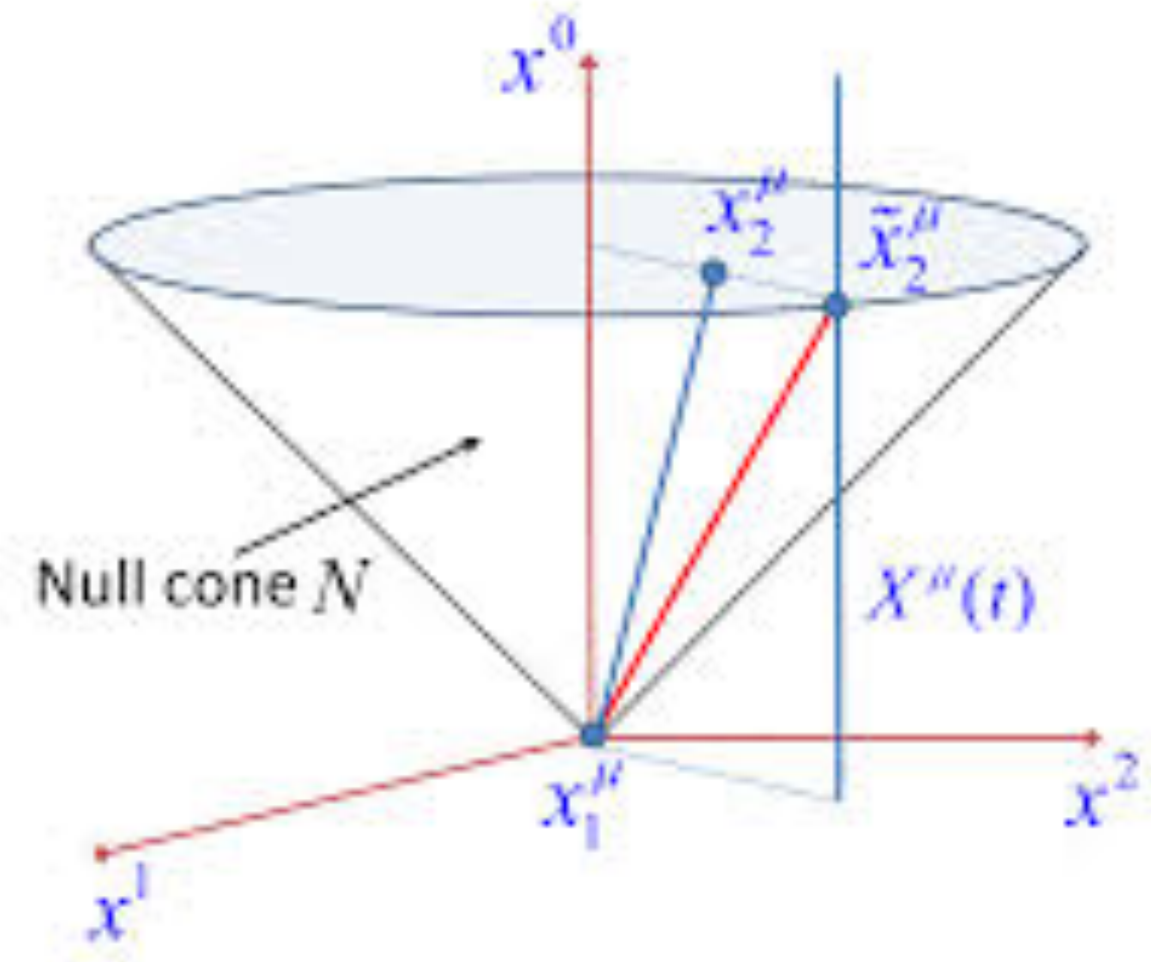
# Bicharacteristic Rays

The conformal metric does not determine the bicharacteristics: We may multiply ${}^{c}k^{\mu}$ by any non-vanishing scalar function $e^{\varphi}$ and define

$$k^{\mu} = e^{\varphi}\, {}^{c}k^{\mu} .$$

To define periods and wavelengths, we need to introduce the scalar volume function $e^{\varphi}$ and define the metric $g^{\mu\nu} = e^{\varphi}\, {}^{c}g^{\mu\nu}$ .

# Temporal and Spatial Projections of a Null Vector



# Symplectic Geometry

**At this level of *symplectic geometry*, wave fronts have Phases, and rays have periods (frequencies) and wave lengths relative to some frame of reference.**

**Given an orthonormal basis in this f.o.r., we project $k^{\mu}$**

**onto the time-like vector to get the period $T$:**

$$k^{\mu} e^{(t)}{}_{\mu} = cT$$

**onto a space-like vector to get the wave length $\lambda$:**

$$k^{\mu} e^{(s)}{}_{\mu} = \lambda.$$

**Since $k^{\mu}$ is the diagonal of a square, the two must be numerically equal: $cT = \lambda$. And since $T = 1/\nu$ , $\lambda\nu = c$.**

# Highest Level of Abstraction: Contact geometry

One can formulate:

Huygen's principle,

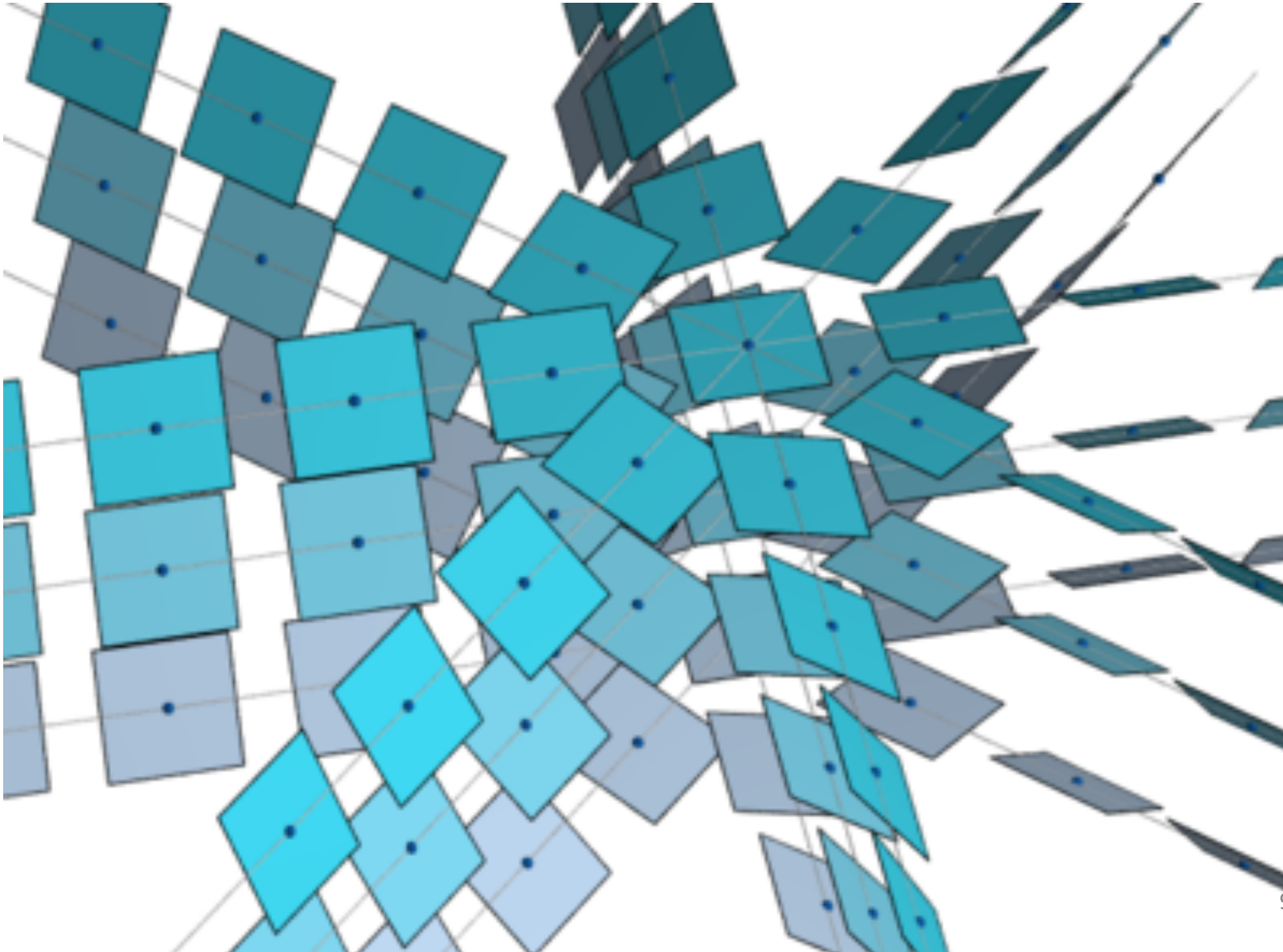
characteristic wave fronts,

↕

Fermat's principle,

bicharacteristic rays

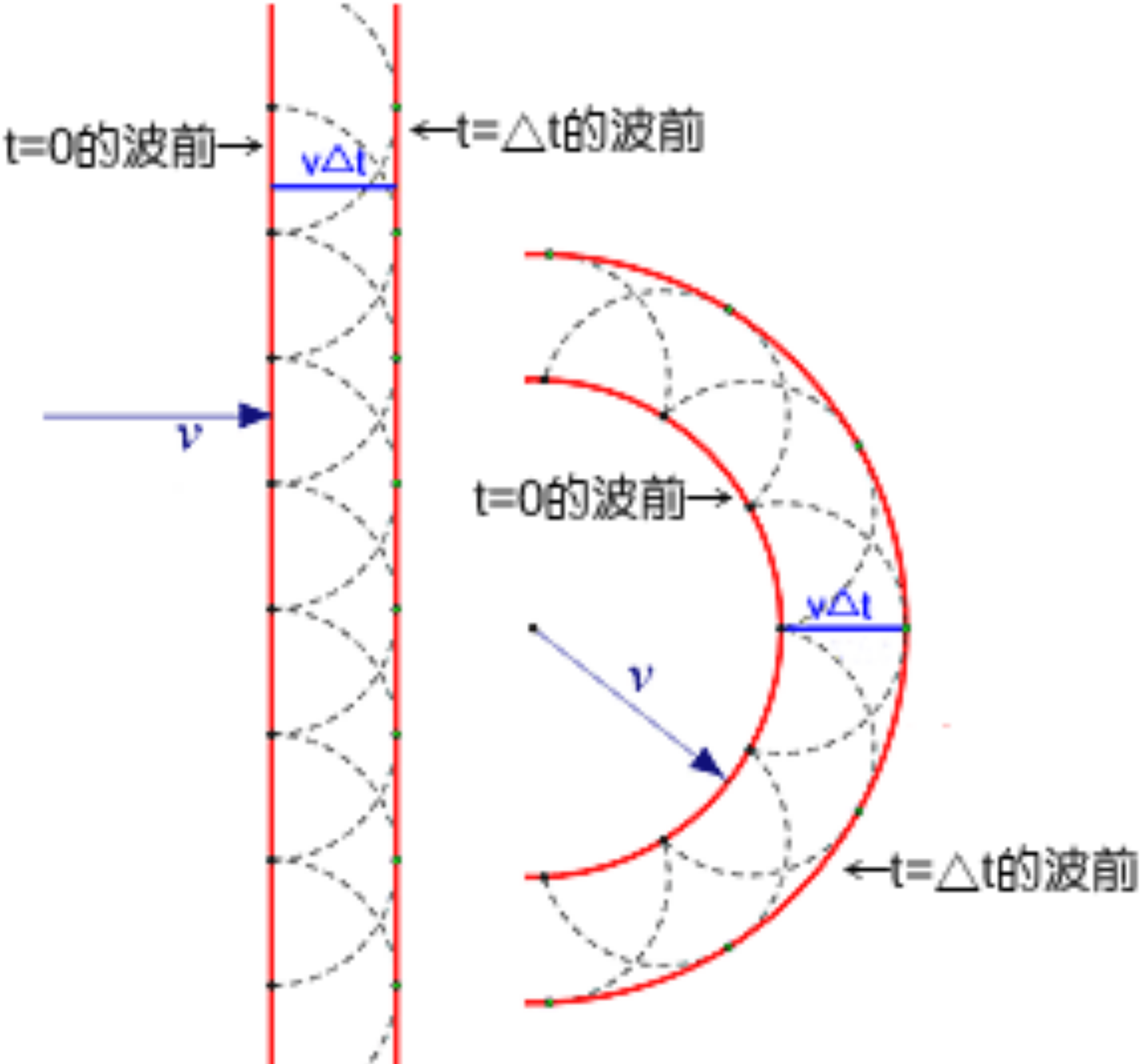
# Contact Structure



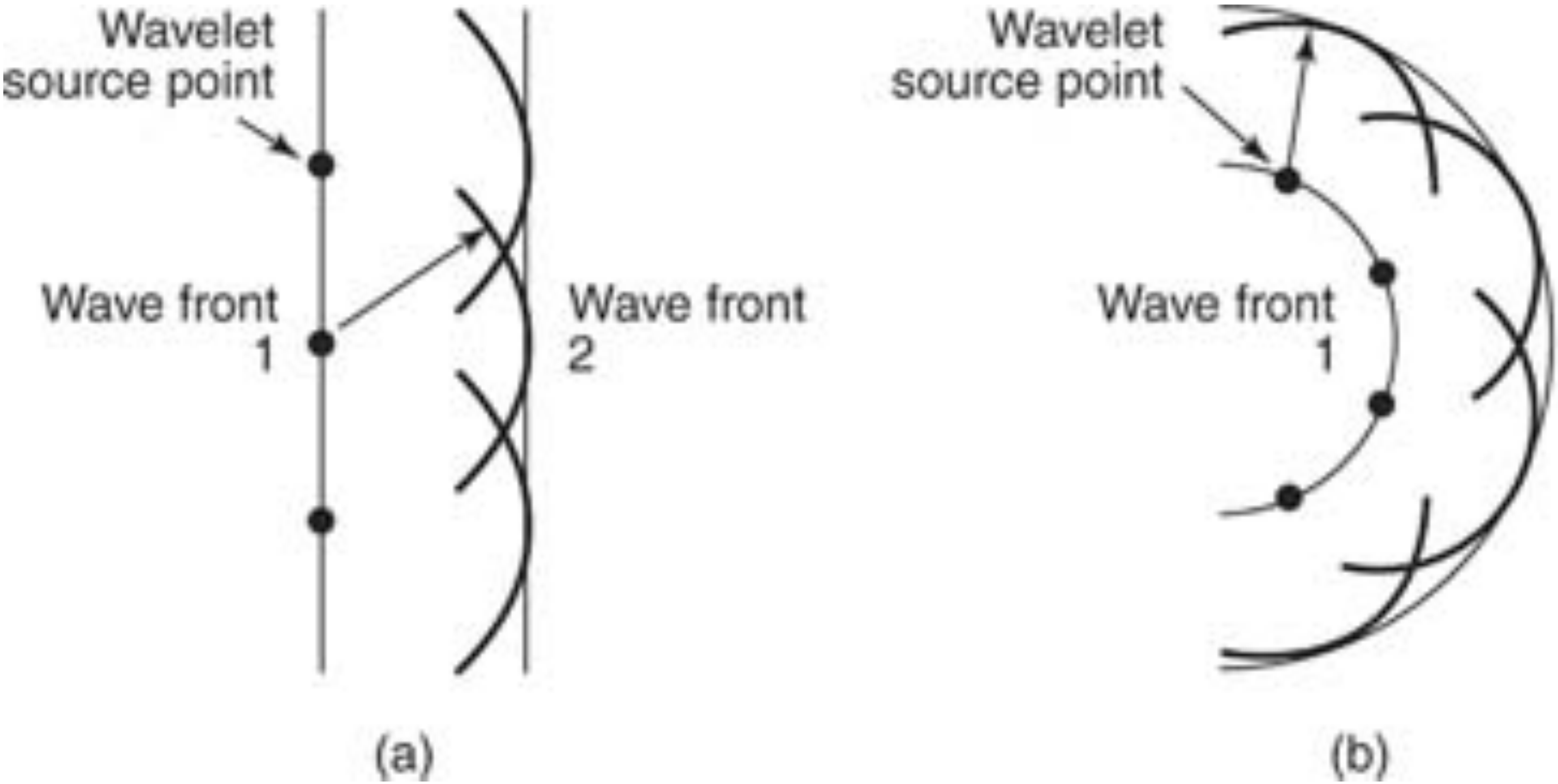
# Mechanics *vs* Optics

- Mechanics: Spacelike Hyper-surface– the normal leads from one hypersurface to the next
- Optics: Null Hypersurface– the normal lies on the hypersurface, a ray leads from one wave front to the next

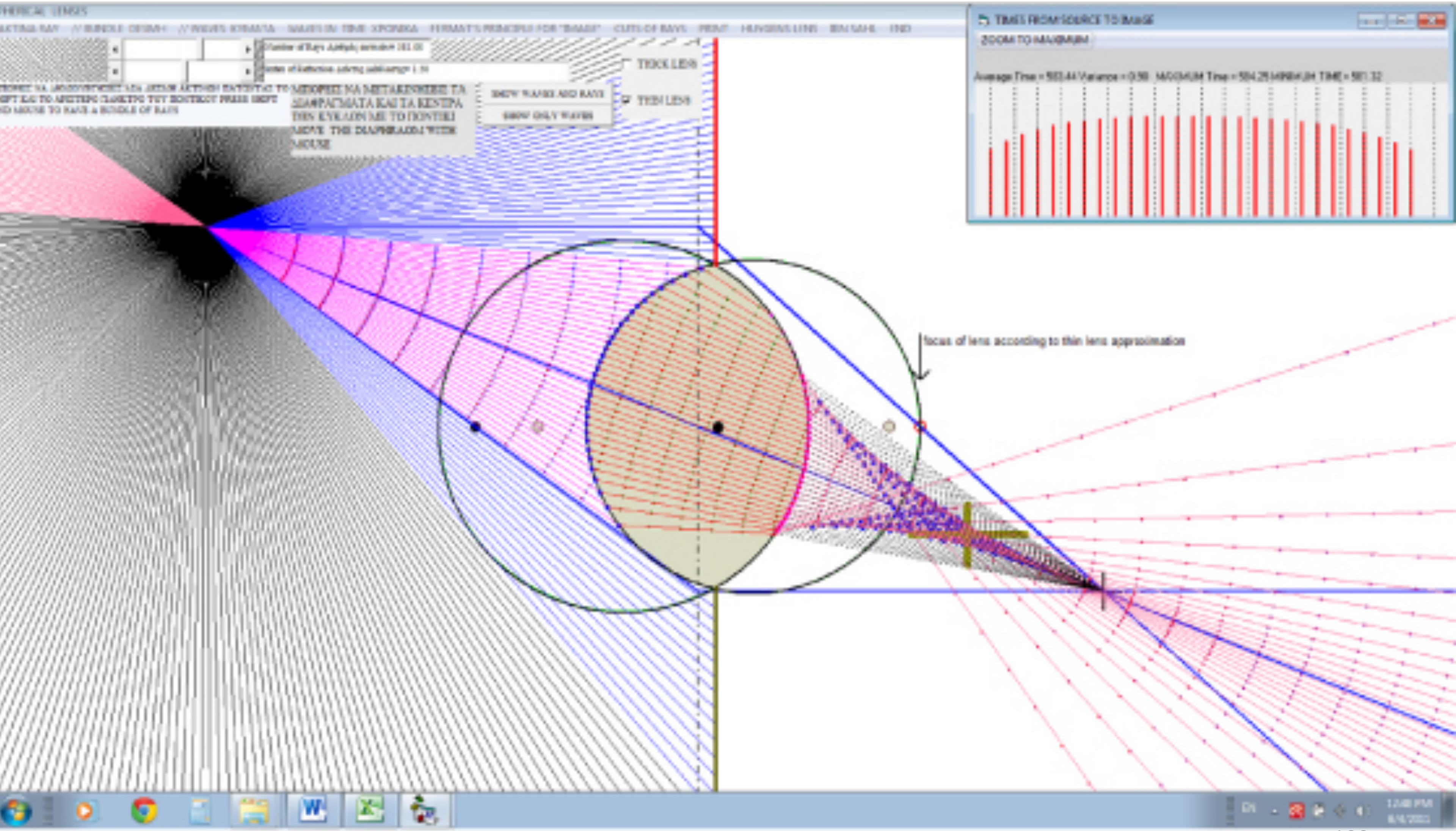
# Huygens' Principle



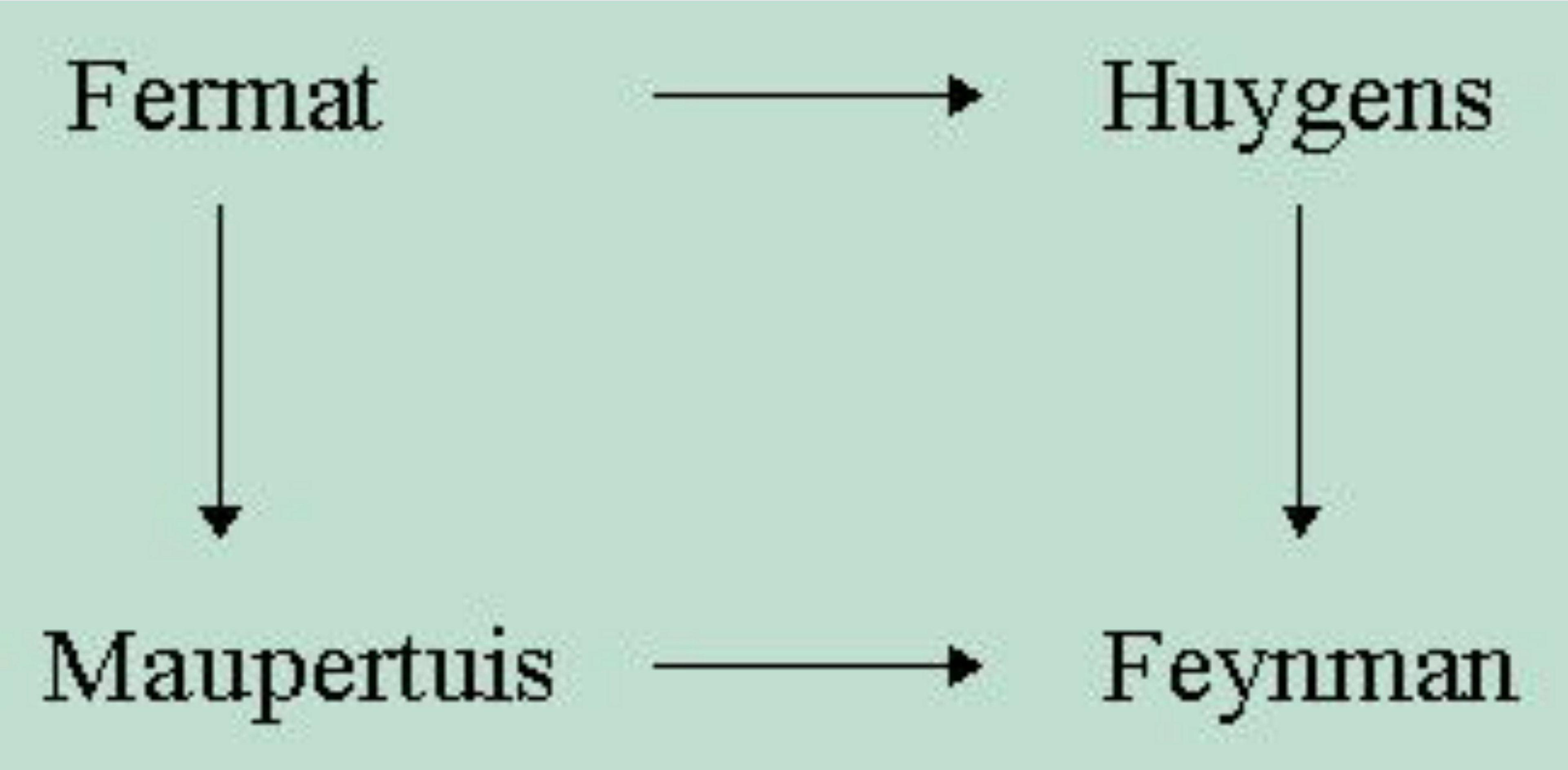
# Huygens' Principle



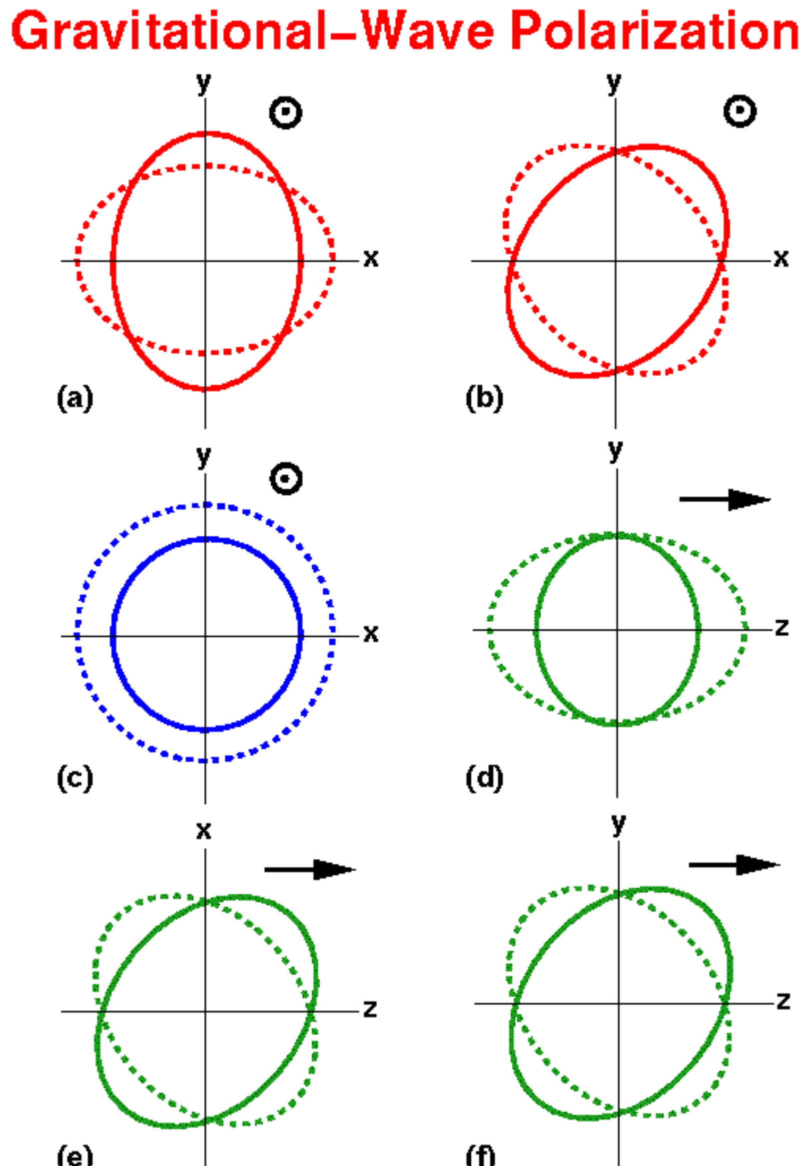
# Huygen’s & Fermat’s Principles



# Relations Between The Four Principles



# Far Field- Pure Radiation



# Far Field- Pure Radiation

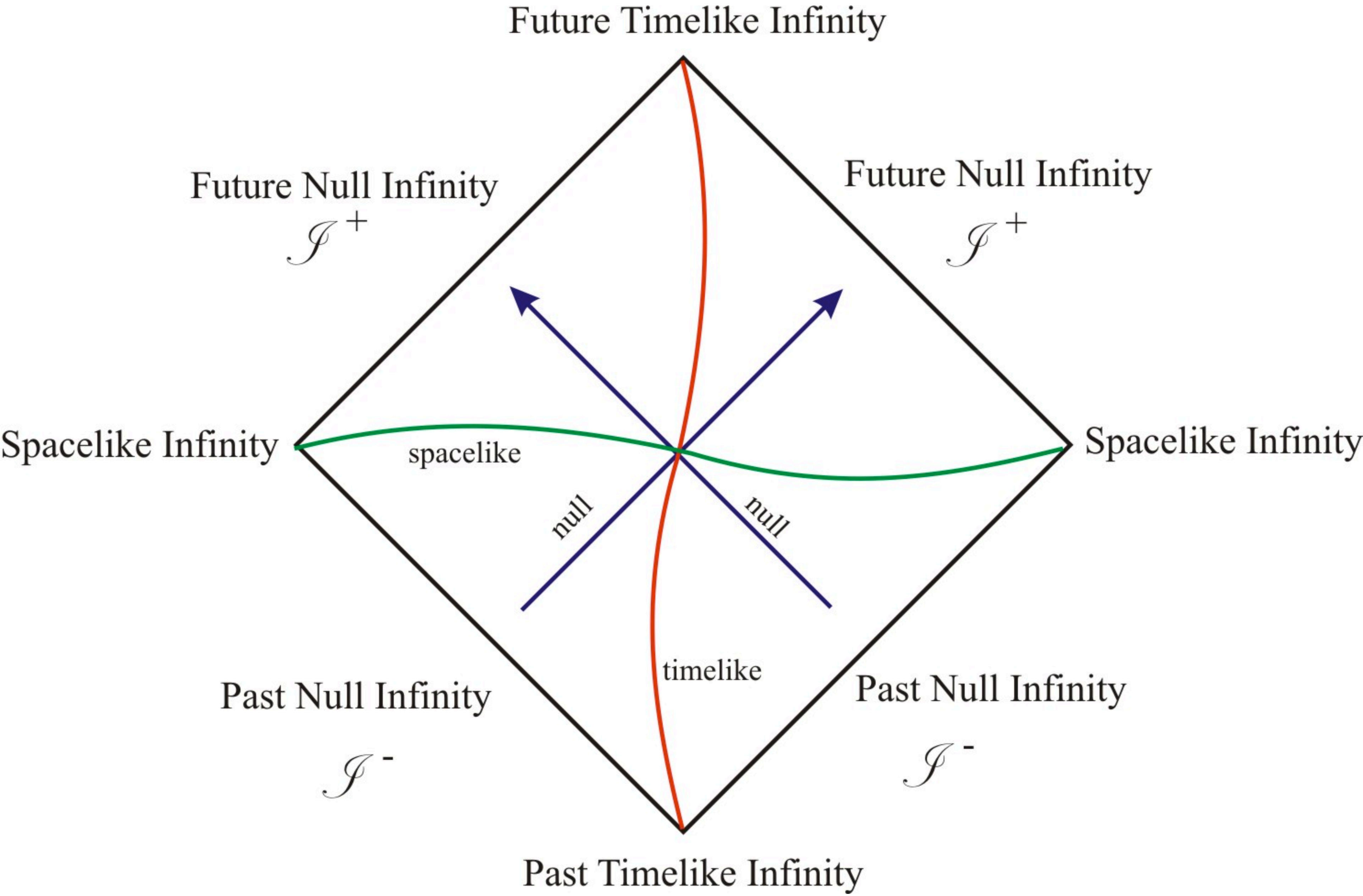
**Conformal Metric**: Conformal curvature tensor

**Weak Field Approx' n**: Assume the gravitational field is **weak**, so we can treat it as a **perturbation** around a **flat Minkowski** space-time metric

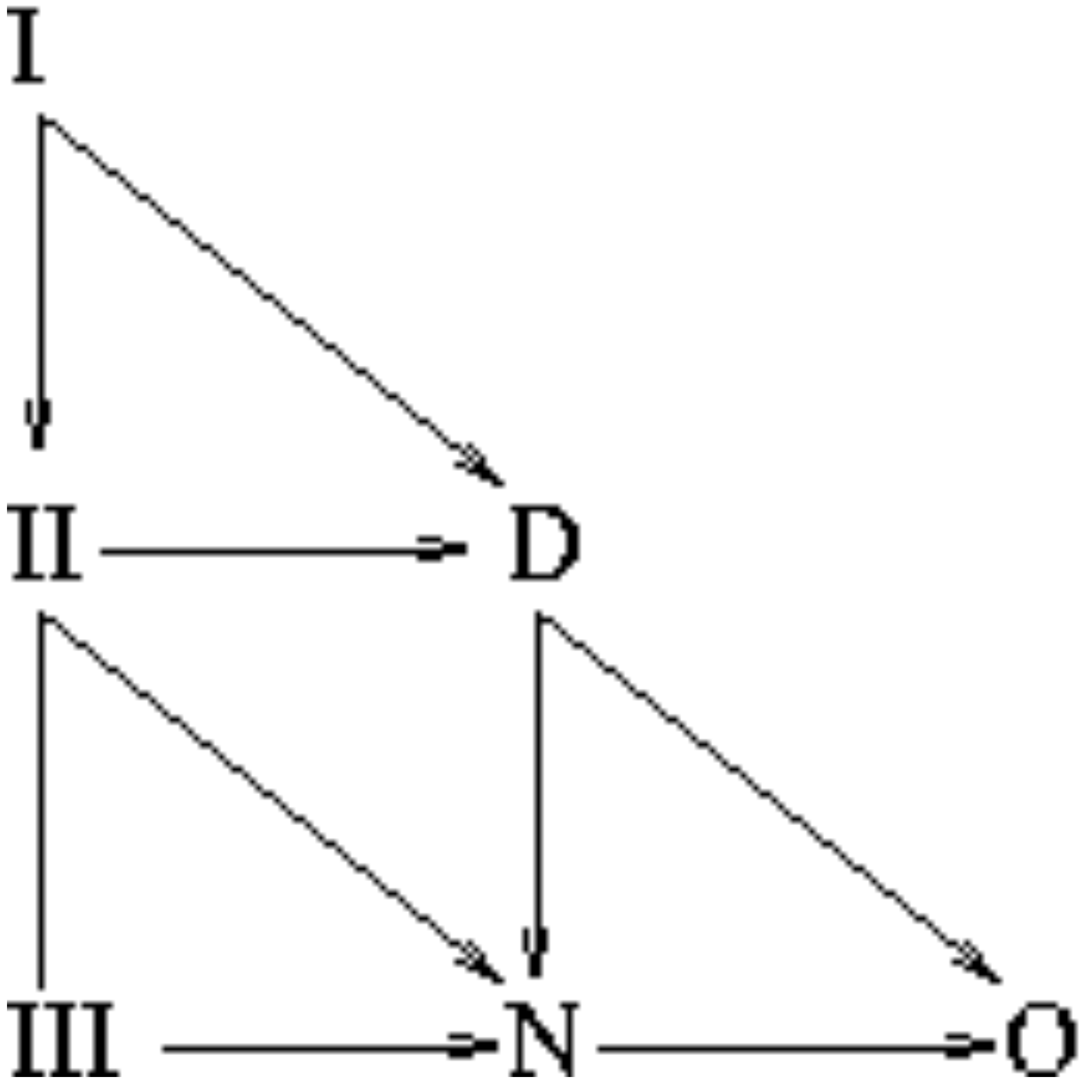
# Asymptotic Quantization

The Bondi-Metzner-Sachs field of any bounded radiating source asymptotically approaches the "free" gravitational radiation field, which is represented by the conformal structure at null infinity as defined by Penrose. This field has been quantized by techniques anticipated by Komar and developed in full detail by Ashtekar. The resulting asymptotic gravitons are representations of the Bondi-Metzner-Sachs group.

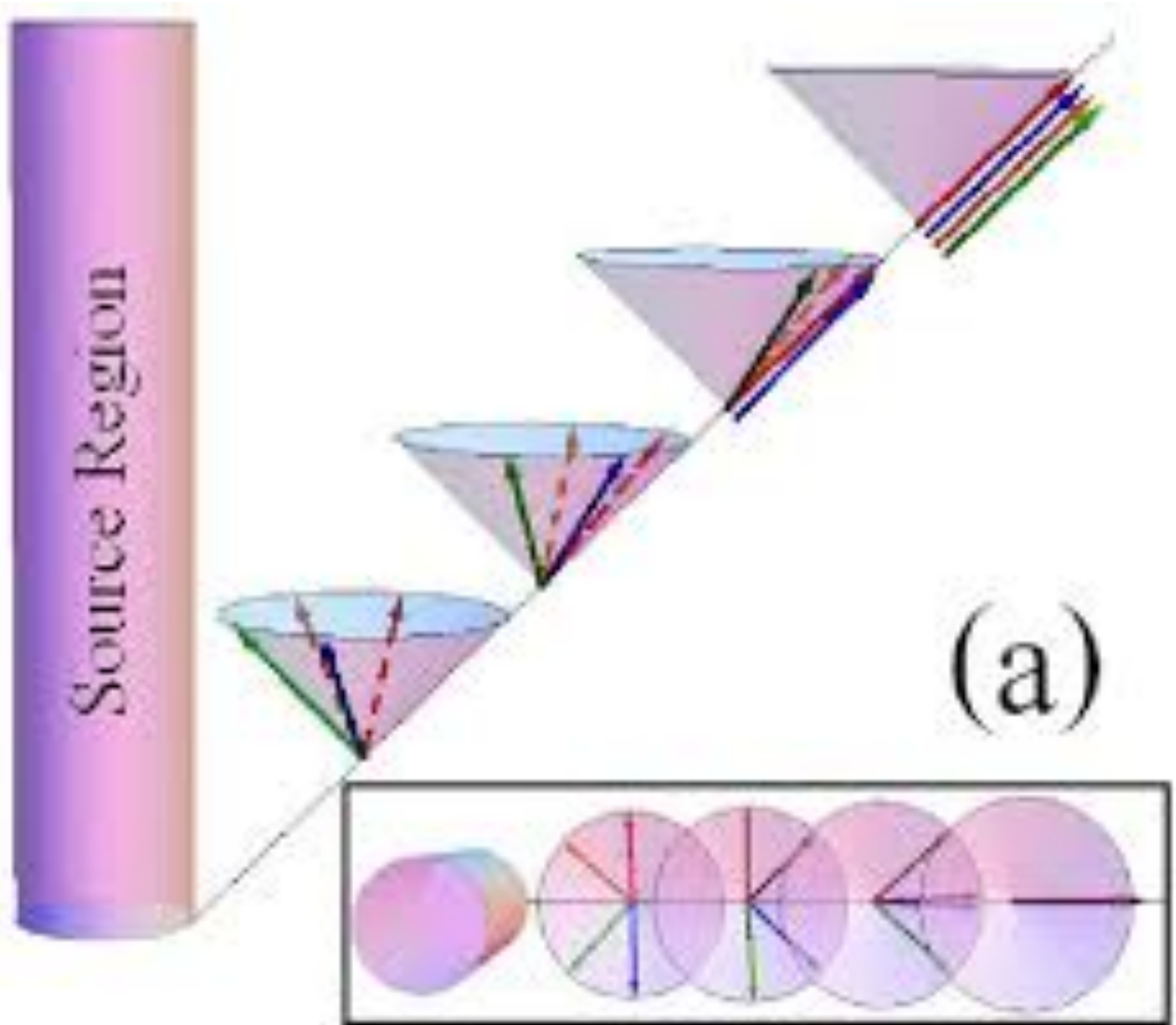
# Abhay Ashtekar:
# *Asymptotic Quantization*


Future Timelike Infinity
Future Null Infinity
$\mathscr{I}^+$
Future Null Infinity
$\mathscr{I}^+$
Spacelike Infinity
spacelike
Spacelike Infinity
null
null
timelike
Past Null Infinity
$\mathscr{I}^-$
Past Null Infinity
$\mathscr{I}^-$
Past Timelike Infinity

# Petrov-Pirani Classification



# The Peeling Theorem



# Asymptotic Quantization

**One can treat "pure" radiation fields independently of their sources; one needs only conformal structures to formulate the results mathematically, interpret them physically, and describe procedures for their measurement. it should thus be possible to quantize these fields directly. The homogeneous Maxwell equations are conformally invariant, so one should also be able to treat interacting "free" electromagnetic and gravitational fields by conformal techniques.**

# Near Field– Sources

**Projective Connection**: Projective curvature tensor

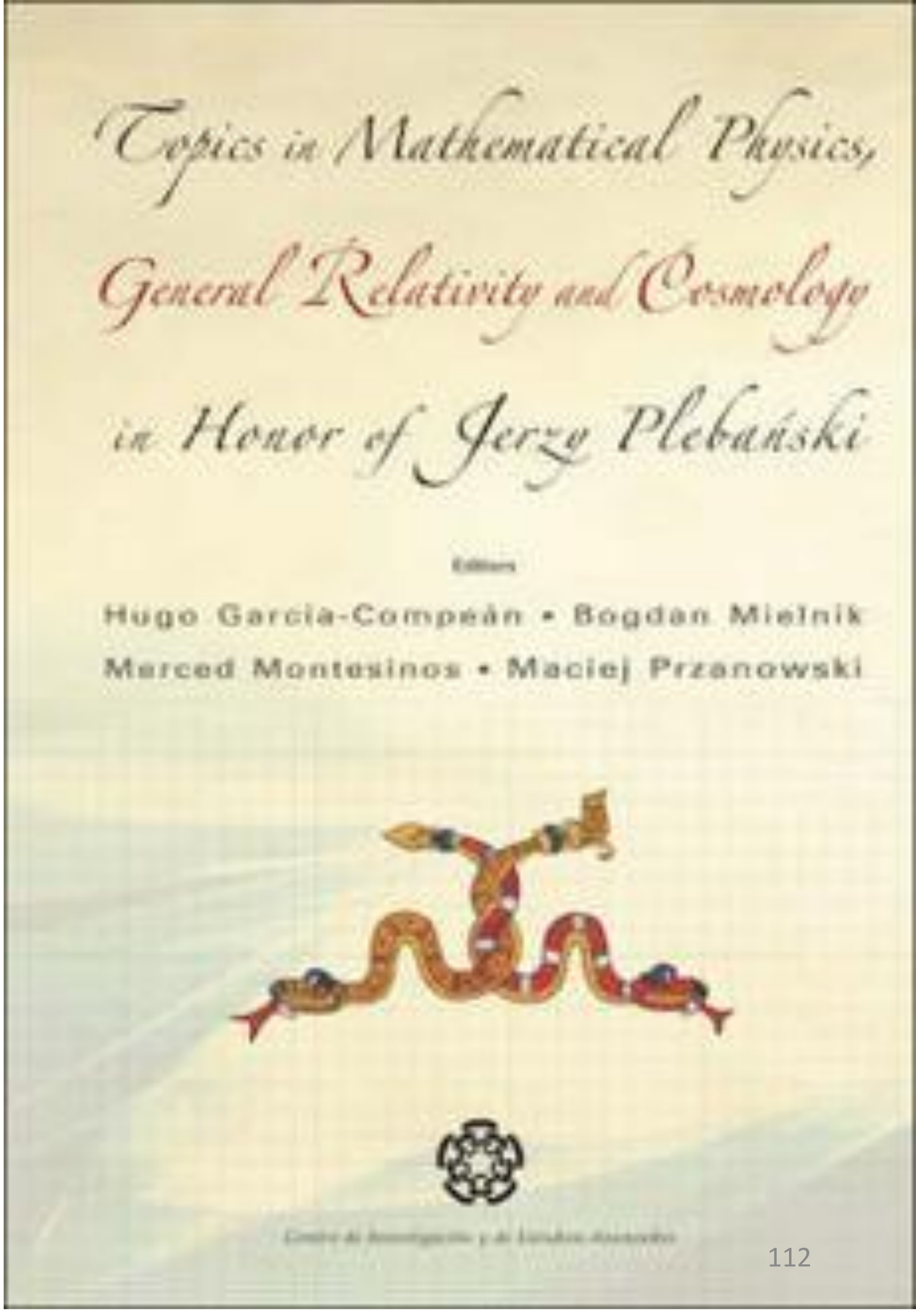
**Post-Newtonian Approx' n**: assume **motions** of sources are **slow**, so that we can treat the gravitational field as a **perturba-tion** about a **Newtonian solution**.

# Newtonian Theory-Modern Version

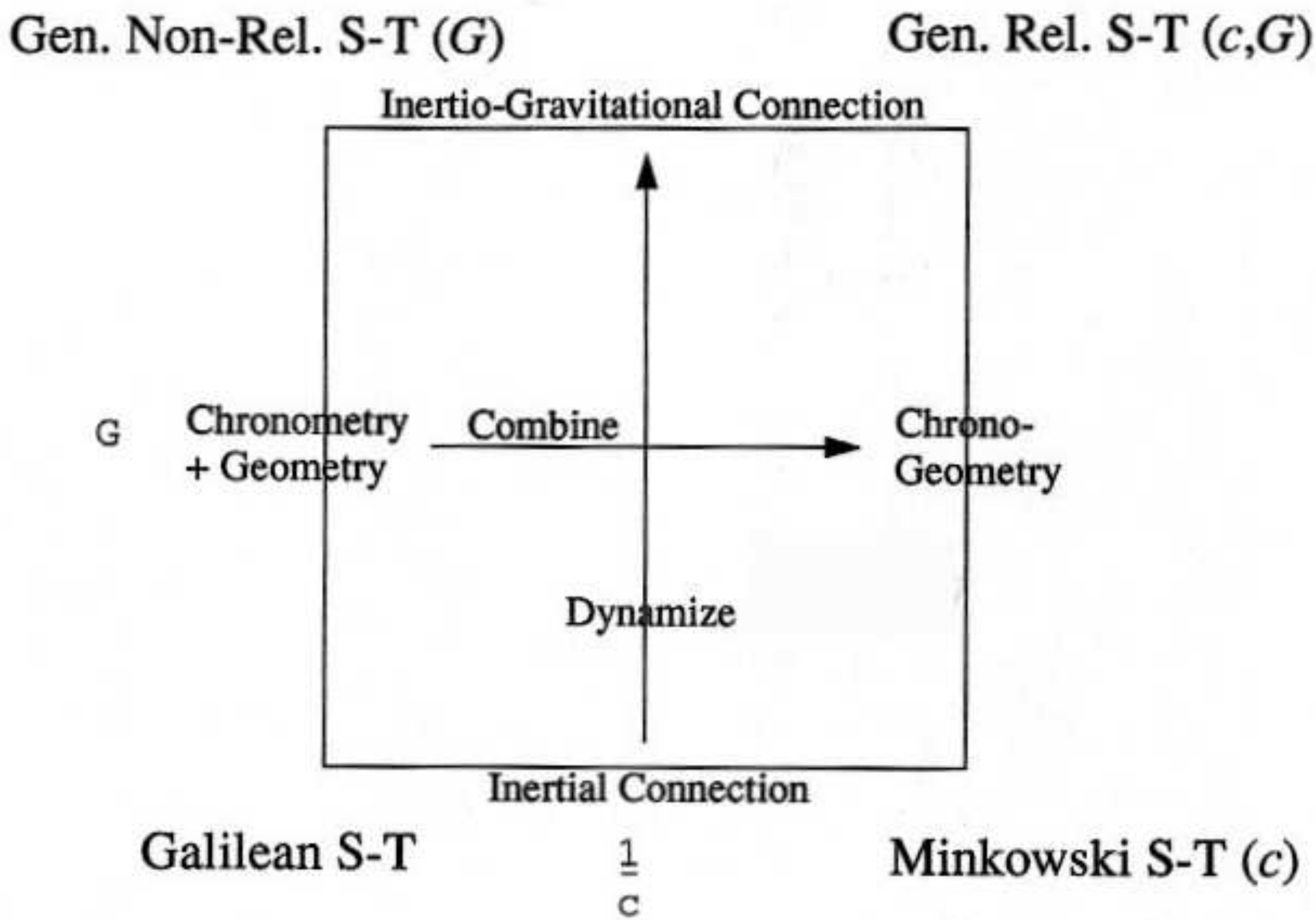
Even at the Newtonian level, gravitation is not an external force acting on bodies, but a modification of the hitherto fixed inertial structure of space-time, which now becomes a dynamical structure, the inertio-gravitational field.

“*Einstein’s Intuition & the Post-Newtonian Approximation*”

**(JS, published 2006,based on paper given in 2002)**



# The Bronstein Square



# What is four-dimensional Newtonian gravitation?

[A]lthough … it leads us beyond Newton's original theory, the following definition seems to me to do no violence to the concept of a Newtonian-style gravitational theory: We shall require Newtonian chronometry and geometry and the compatibility conditions between both and the inertio-gravitational connection to hold. In other words, a Newtonian-style theory is one that is based on a Galileian manifold and a compatible affine connection.

# What is four-dimensional Newtonian gravitation?

**Newtonian theory… allows us to go a bit further than traditional Newtonian gravitation theory. … [A]nalysis of the compatibility conditions on the t[etrad] c[oefficients of the] c[onnection] shows that they allow the $\Gamma^{(c)}{}_{(0)(0)}$ to be non-vanishing; this is well known, since they correspond physically to the electric-type Newtonian gravitational field produced by masses at rest, i.e., the $\rho$ term in the $T^{(0)(0)}$ component of the stress-energy tensor-- all that conventional Newtonian theory considers.**

# What is four-dimensional Newtonian gravitation?

**But the compatibility conditions also allow non-vanishing $\Gamma^{(c)}{}_{(0)(b)}$ , which does not seem to have been noted. Physically, these components correspond to a magnetic-type Newtonian gravitational field, produced by moving masses, corresponding to the $\rho v$ or $T^{(0)(i)}$ components and not present in traditional Newtonian theory.**

# What is four-dimensional Newtonian gravitation?

It is then simple to show that a rotating source, by creating such a **magnetic-type Newtonian gravitational field $\Gamma^{(c)}{}_{(0)(b)}$**, drags along the inertial frames with it. So the recent observation of the **dragging of inertial frames by the rotation of the earth** is a confirmation of the existence of a **Newtonian** magnetic-type field. We are still far from being able to observe any GR corrections to this effect.

# What is four-dimensional Newtonian gravitation?

**The compatibility conditions between the dynamic inertio-gravitational connection and the fixed Euclidean geometry on the fixed absolute time hypersurfaces *T* = *const* imply that the trace of the connection vanishes.**

**So it is a projective connection, and it is the projective Ricci tensor that enters the gravitational field equations.**

# What Can UCPR Contribute?

Quantize the near field produced by motion of the source using the projective curvature tensor

Quantize the far radiation field using the conformal curvature tensor

Connect the two by the method of matched asymptotic expansions

# For More Details See

**“Quantum Gravity: A Heretical Vision,” in S. Burra et al. (eds.), *Frontiers of Fundamental Physics and Physics Education Research* (Springer Proceedings in Physics 145, 2014), pp. 149-158,**

**and/or my Power Point:**

# Quantum Gravity: A Heretical Vision

**John Stachel**

**FFP12, Udine 21-23 November 2012**

**You can contact me for comments, questions and copies at**

**john.stachel@gmail.com**

# Current Work in Progress

**The role of the Frölicher-Nijenhuis Bracket in the contravariant metric approach to GR**

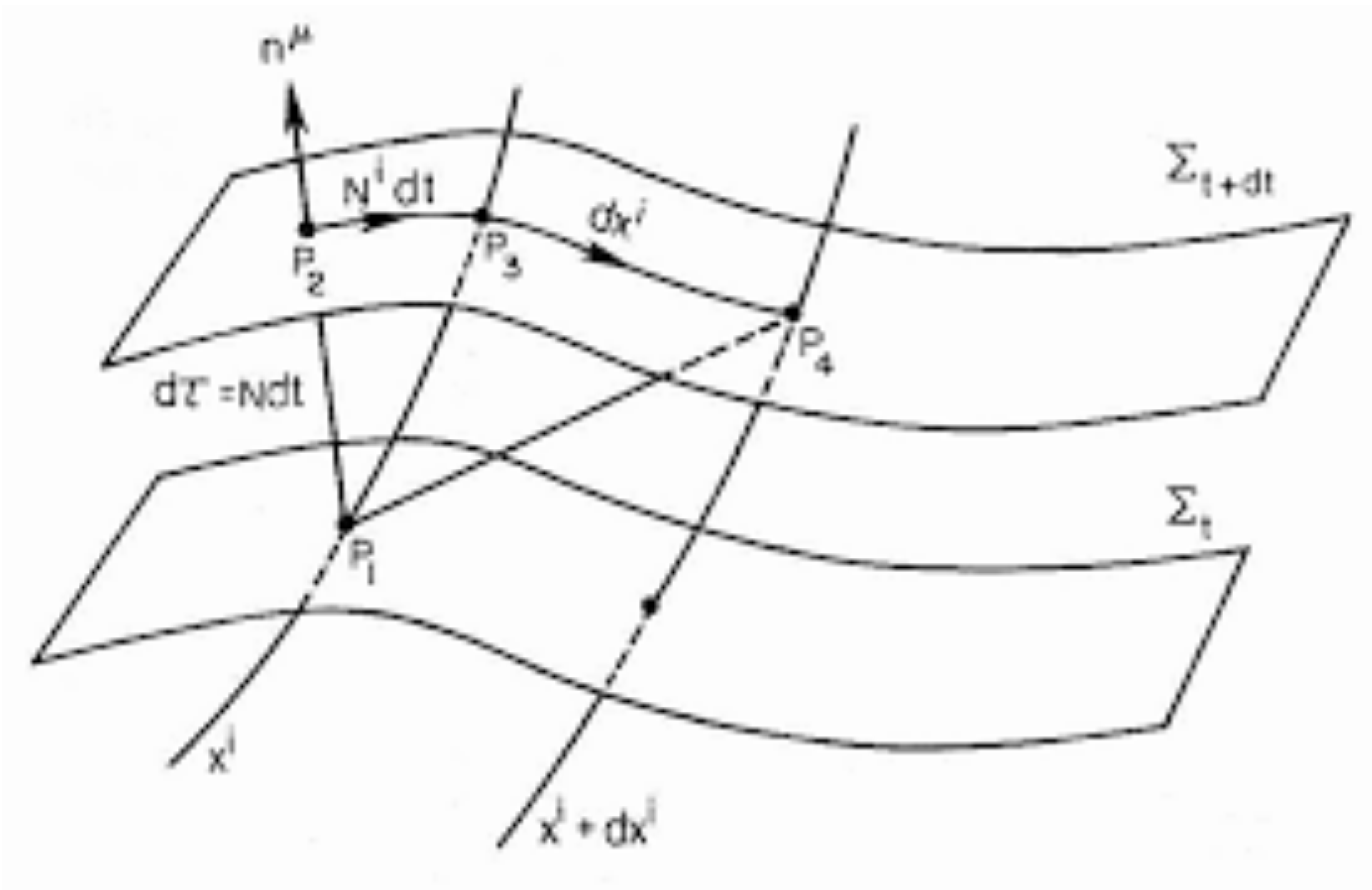
# Cauchy Initial Value Problem

**Starting with the covariant metric $g_{\mu\nu}$ and contravariant vector field $V^{\mu}$(x) defining a fibration of $\mathcal{M}$, one picks an initial hypersurface and drags it by the vector field to produce a foliation of $\mathcal{M}$. One then decomposes $g_{\mu\nu}$ w.r.t. this fibration and foliation, and poses a Cauchy problem with initial data on one hypersurface:**

**first fundamental form $'g_{\mu\nu}$ and**

**second fundamental form $h_{\mu\nu}$, = -1/2$\mathcal{L}_{V}{'g_{\mu\nu}}$.**

# Cauchy Initial Value Problem



# Cauchy Initial Value Problem

By requiring $V$ to be a geodesic normal field $n$, we can show that:

$$\mathcal{L}^2{}_n\,{}'g_{\mu\nu} = 2(h_\mu{}^\rho h_{\rho\nu} - B^\rho{}_\mu B^\sigma{}_\nu R_{\alpha\rho\sigma\beta}\, n^\alpha n^\beta).$$

$B^\rho{}_\mu B^\sigma{}_\nu R_{\alpha\rho\sigma\beta}\, n^\alpha n^\beta$ is determined by six field equations $B^\rho{}_\mu B^\sigma{}_\nu R_{\rho\sigma} = 0$ together with ${}'g_{\mu\nu}$ and $h_{\mu\nu}$ ; and the Gauss-Codazzi equations together with ${}'g_{\mu\nu}$ and $h_{\mu\nu}$ determine all other components of the Riemann tensor.

# Is There a *Dual* Initial Value Problem?

**But suppose, starting with the contravariant metric $g^{\mu\nu}$ and covariant vector field $\omega_\mu$ (x) defining a foliation of $\mathcal{M}$, one picks an initial curve. Can one drag it by the covector field to produce a fibration of $\mathcal{M}$?**

**And if so, can one then decompose $g^{\mu\nu}$ w.r.t. the fibration and foliation, and pose an initial value problem for the fiber curve's part of $g^{\mu\nu}$?**

**And if so what takes the place of the Lie derivative?**

# Yes! There is a Dual Approach

Take **three commuting** vector fields $W_{(A)}{}^{\mu}$ that span each of the hypersurfaces of the foliation:

$$\omega_\mu W_{(A)}{}^{\mu} = 0,$$

and drag $V^\nu$ with each of them:

$$\mathcal{L}_{W(A)} V^\nu.$$

We showed this preserves the condition

$$\omega_\mu V^\mu = 1.$$

# Yes! There is a Dual Approach

Since the three $W_{(A)}{}^{\mu}$ commute, the fiber created by dragging the initial fiber to any other point will be independent of the path taken between the two points.

So we have produced a **fibration** of $\mathcal{M}$. We can now show that

$$\omega_{\mu}\mathcal{L}_{W(A)}V^{\nu} = [\omega_{\mu}\, W_{(A)},\, V]^{\nu}{}_{FN}\,.$$

Here, the **Frölicher-Nijenhuis Bracket** takes the place of the Lie derivative

# The Frölicher-Nijenhuis Bracket

We can show that, for a vector-valued one-form $\omega_\mu W^\nu$, and a vector field $V^\kappa$, the Frölicher–Nijenhuis bracket is:

$$[\omega_\mu W, V]^\nu{}_{FN} = \omega_\mu \mathcal{L}_W V^\nu - (\mathcal{L}_V \omega_\mu) W^\nu.$$

If

$$\omega_\mu V^\mu = 1 \quad \text{and} \quad \omega_\mu W^\mu = 0,$$

this reduces to

$$[\omega_\mu W, V]^\nu{}_{FN} = \omega_\mu \mathcal{L}_W V^\nu$$

NATURAL OPERATIONS IN DIFFERENTIAL GEOMETRY

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[illegible]

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Faculty of Science, Masaryk University
[illegible]

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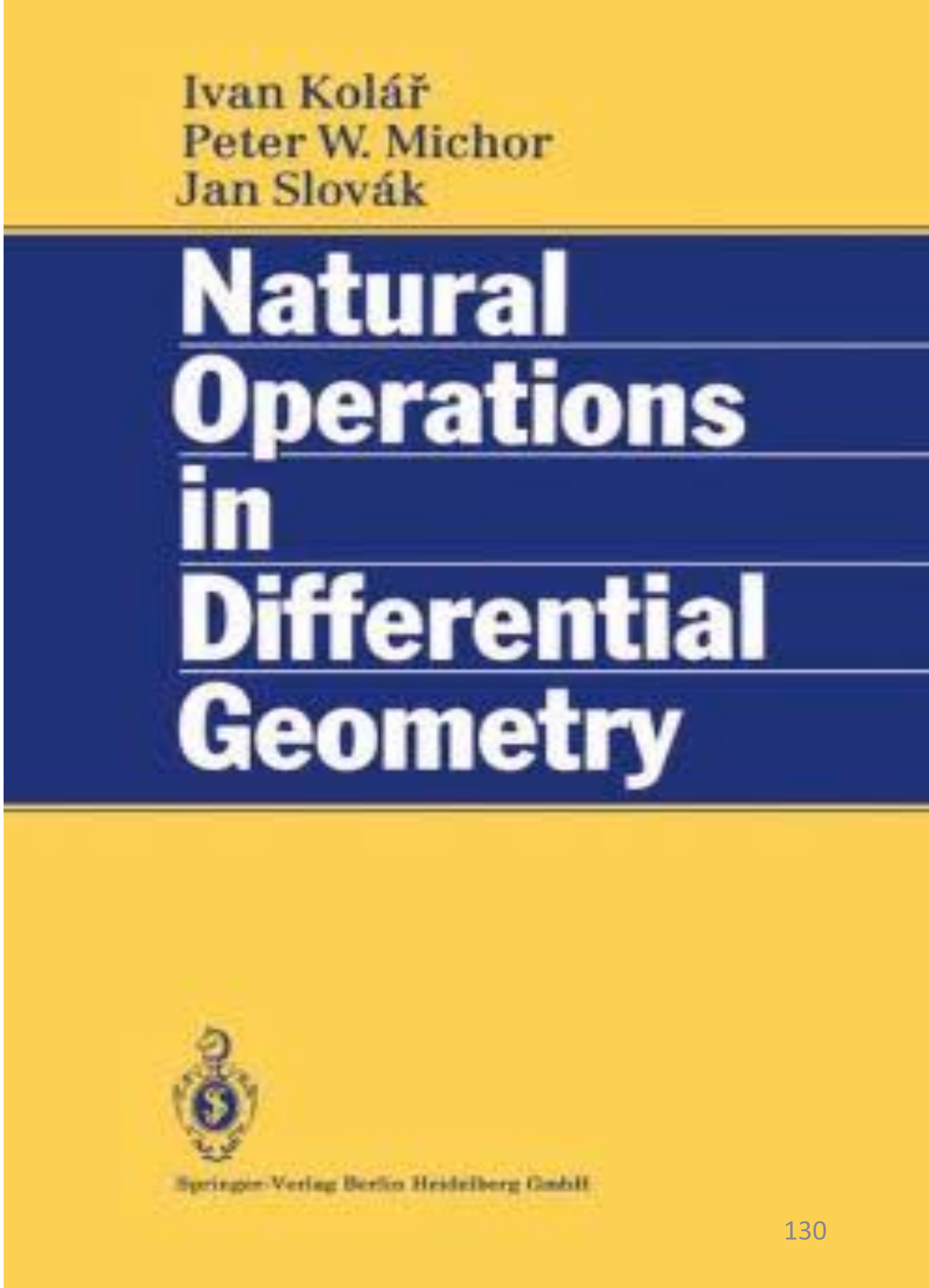

# The Frölicher-Nijenhuis Bracket

"Here we present the The Frölicher-Nijenhuis Bracket (a natural extension of the Lie bracket from vector fields to vector valued differential forms) as one of the basic structures of differential geometry and we base nearly all treatment of curvature and Bianchi identities on it."

# The Frölicher-Nijenhuis Bracket

**"This allows us to present the concept of a connection first on general fiber bundles (without structure group), with curvature, parallel transport and Bianchi identity, and only then add G-equivariance as a further property for principal fiber bundles."**

# The Frölicher-Nijenhuis Bracket

**"We think that in this way the under-lying geometric ideas are more clearly understood by the novice than in the traditional approach, where too much structure at the same time is rather confusing."**


Vielen Dank
für Ihre
Aufmerksamkeit