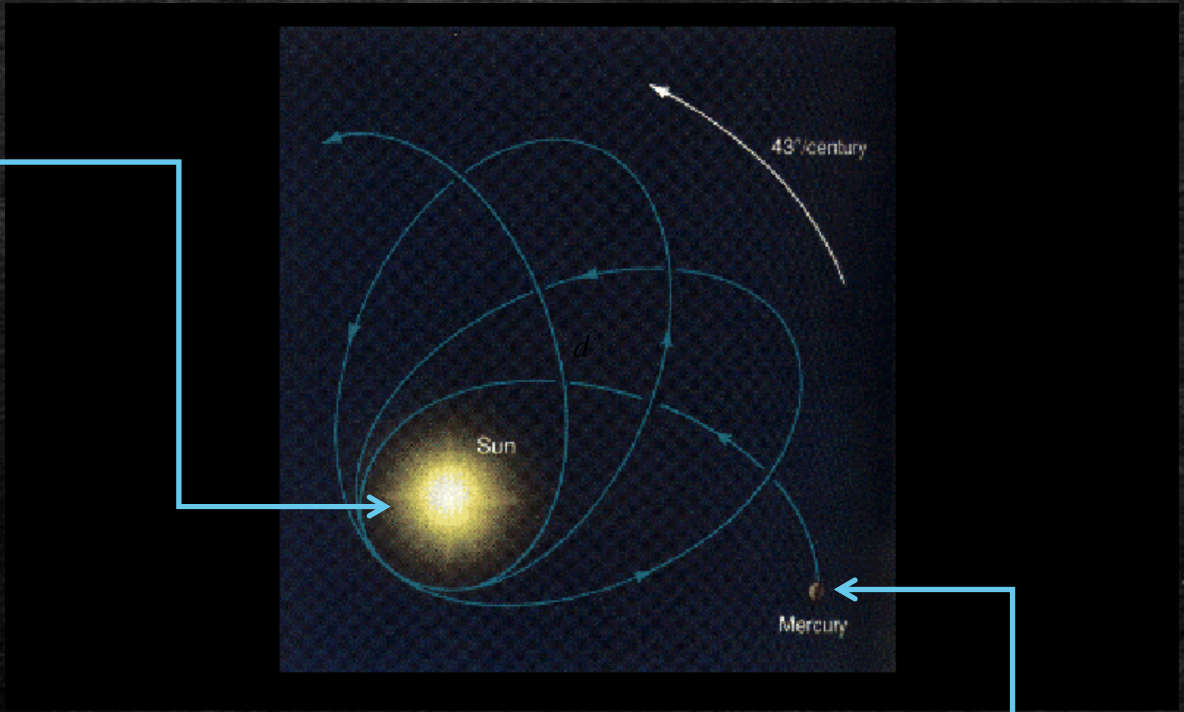
On different approaches to the problem of motion in General Relativity

Dennis Lehmkuhl, Einstein Papers Project, Caltech

2 equations are at the heart of GR

The Einstein field equations:

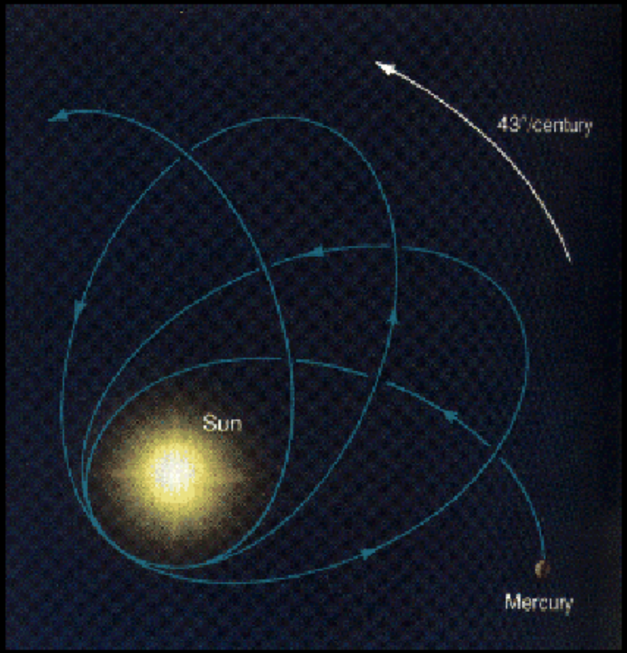
$$G_{\mu\nu} = \kappa T_{\mu\nu}$$



The geodesic equation:

$$\frac{d^2 x_\tau}{ds^2} + \Gamma^\tau{}_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0$$

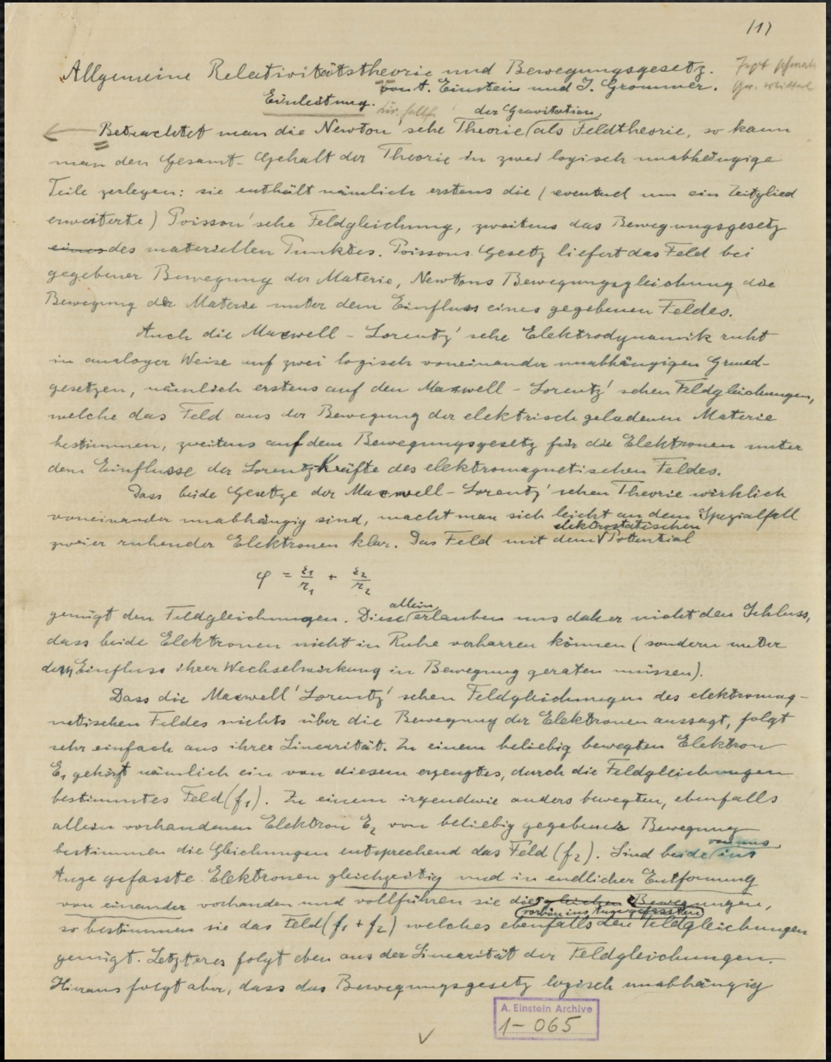
The problem of motion 1916 and 1927

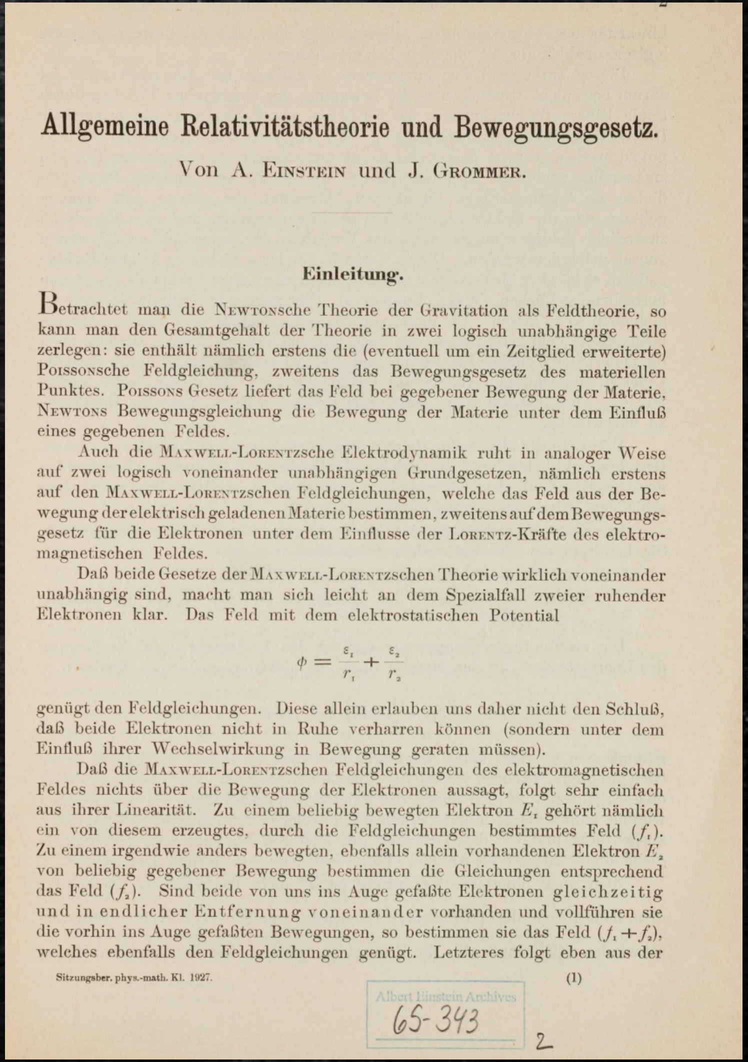


Is it the case that all we need to know to find out how Mercury is moving is the gravitational field of the sun?

Einstein 1916: No.
Einstein 1927: Yes.

The Einstein-Grommer paper of 1927





Allgemeine Relativitätstheorie und Bewegungsgesetz.

Von A. Einstein und J. Grommer.

Einleitung.

Betrachtet man die Newtonsche Theorie der Gravitation als Feldtheorie, so kann man den Gesamtgehalt der Theorie in zwei logisch unabhängige Teile zerlegen: sie enthält nämlich erstens die (eventuell um ein Zeitglied erweiterte) Poissonsche Feldgleichung, zweitens das Bewegungsgesetz des materiellen Punktes. Poissons Gesetz liefert das Feld bei gegebener Bewegung der Materie, Newtons Bewegungsgleichung die Bewegung der Materie unter dem Einfluß eines gegebenen Feldes.

Auch die Maxwell-Lorentzsche Elektrodynamik ruht in analoger Weise auf zwei logisch voneinander unabhängigen Grundgesetzen, nämlich erstens auf den Maxwell-Lorentzschen Feldgleichungen, welche das Feld aus der Bewegung der elektrisch geladenen Materie bestimmen, zweitens auf dem Bewegungsgesetz für die Elektronen unter dem Einflusse der Lorentz-Kräfte des elektromagnetischen Feldes.

Daß beide Gesetze der Maxwell-Lorentzschen Theorie wirklich voneinander unabhängig sind, macht man sich leicht an dem Spezialfall zweier ruhender Elektronen klar. Das Feld mit dem elektrostatischen Potential

$$\phi = \frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2}$$

genügt den Feldgleichungen. Diese allein erlauben uns daher nicht den Schluß, daß beide Elektronen nicht in Ruhe verharren können (sondern unter dem Einfluß ihrer Wechselwirkung in Bewegung geraten müssen).

Daß die Maxwell-Lorentzschen Feldgleichungen des elektromagnetischen Feldes nichts über die Bewegung der Elektronen aussagt, folgt sehr einfach aus ihrer Linearität. Zu einem beliebig bewegten Elektron E_1 gehört nämlich ein von diesem erzeugtes, durch die Feldgleichungen bestimmtes Feld (f_1). Zu einem irgendwie anders bewegten, ebenfalls allein vorhandenen Elektron E_2 von beliebig gegebener Bewegung bestimmen die Gleichungen entsprechend das Feld (f_2). Sind beide von uns ins Auge gefaßte Elektronen gleichzeitig und in endlicher Entfernung voneinander vorhanden und vollführen sie die vorhin ins Auge gefaßten Bewegungen, so bestimmen sie das Feld $(f_1 + f_2)$, welches ebenfalls den Feldgleichungen genügt. Letzteres folgt eben aus der

2

Outline

1. Einstein and Grommer on the three ways of relating field equations and equations of motion
2. Does $T_{\mu\nu}$ belong to General Relativity proper?
3. The Einstein-Rainich correspondence: singularities and non-linearity
4. The Einstein-Grommer approach: from vaccuum field equations to equations of motion
5. Reflections on the Einstein-Grommer approach
6. Conclusion

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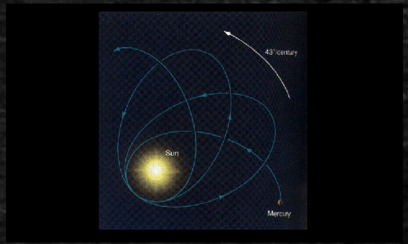
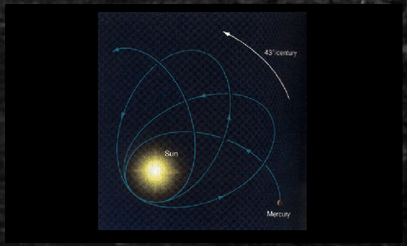
Einstein and Grommer on three ways of relating field equations and equations of motion



1. Newtonian: Poisson's equation for the dynamics of gravitational fields and the equation of motion for particles subject to a gravitational field have to be postulated independently.
2. Maxwellian: Maxwell's equations for the dynamics of the electromagnetic field and the Lorentz' equation of motion for particles subject to an electromagnetic field have to be postulated independently; but the electromagnetic field enters the equations of motion.
3. The new way: In a non-linear theory like GR, there is a chance that the equations of motion are so strongly constrained by the field equations that they actually follow from them.

Einstein and Grommer’s three ways applied to GR.



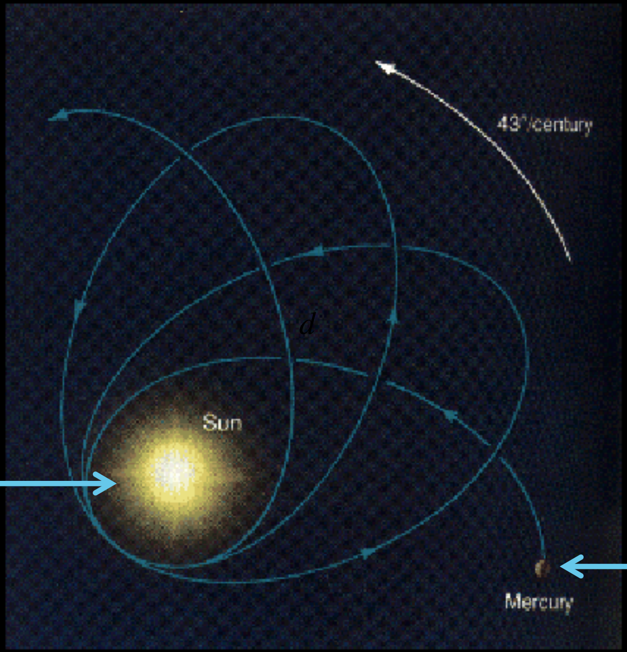
1. Newtonian: The gravitational field equations of empty space and the equations of motion for material particles (the law of geodesic motion) are postulated independently.

The Newtonian way applied to GR

The Einstein field equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

$$R_{\mu\nu} = 0$$



The geodesic equation:

$$\frac{d^2 x_\tau}{ds^2} + \Gamma^\tau{}_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0$$

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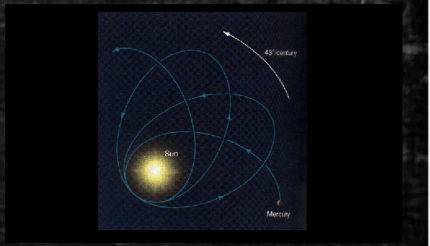


2. Maxwellian: ``The second approach complements the field law by introducing the energy tensor of matter [...] The energy tensor $T_{\mu\nu}$ must be expressed in terms of some (continuous) fields, and the equations determining the behaviour of the latter have to be found; only then the theory is complete."

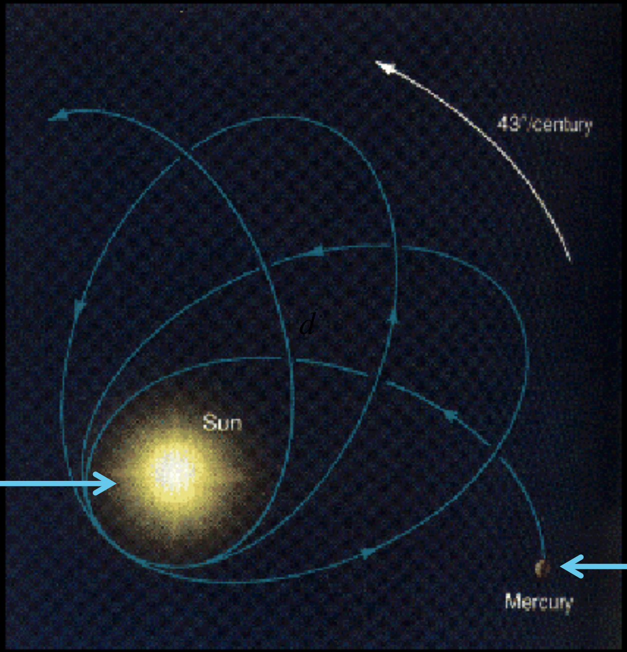
The Maxwellian way applied to GR

The Einstein field equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Energy momentum conservation:

$$\nabla^{\mu} T_{\mu\nu} = 0$$



The geodesic equation:

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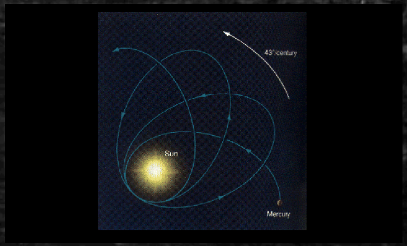
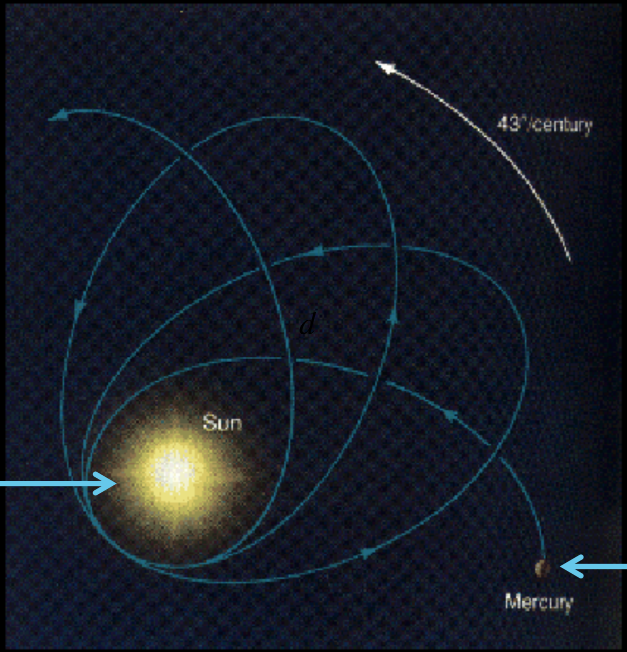


3. The new way: In a non-linear theory like GR, the field of one body described by a solution to the field equations strongly constrains the field another body can have, if the composite system is supposed to be a solution to the field equations, too. Ideally, the constraining is so tight that it amounts to determining the field of the second body.

The new way applied to GR

The Einstein field equations:

$$R_{\mu\nu} = 0$$



- BUT there is a price to pay: matter is represented by singularities.

The geodesic equation:

$$\frac{d^2 x_\tau}{ds^2} + \Gamma^\tau{}_{\mu\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0$$

Two questions now:

a.) Why did Einstein prefer the third way over the second way? Most attempts at deriving the geodesic equation from the field equations before and after went via the energy-momentum tensor, and avoided singularities.

b.) How did the Einstein-Grommer approach come about? What changed between 1916 and 1927?

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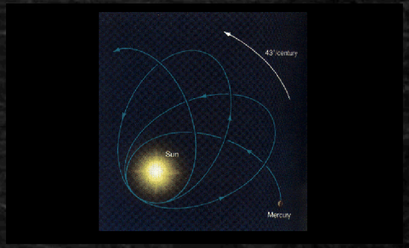
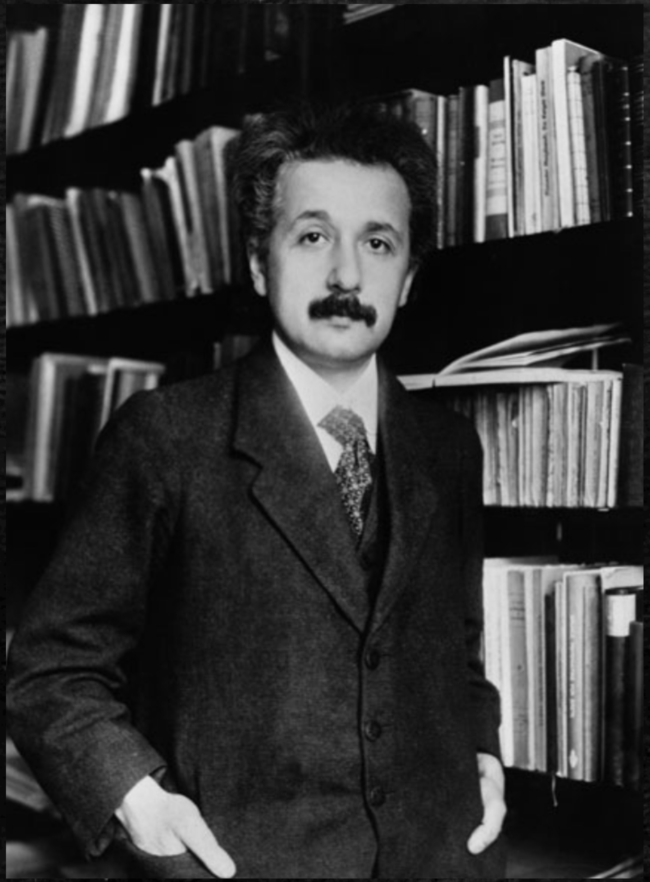
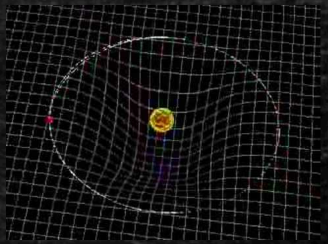
Marble and Wood

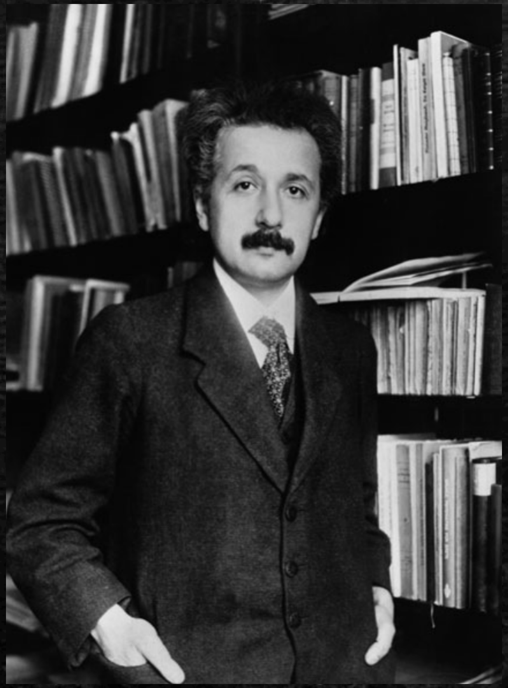




``[GR] is sufficient --- as far as we know --- for the observation of the observed facts of celestial mechanics. But it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low-grade wood (right side of equation). The phenomenological representation of matter is, in fact, only a crude substitute for a representation which would do justice to all known properties of matter. " Einstein (1936)

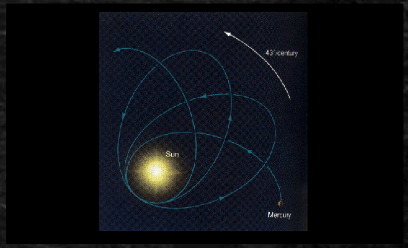
Einstein and the geometric interpretation, 1925

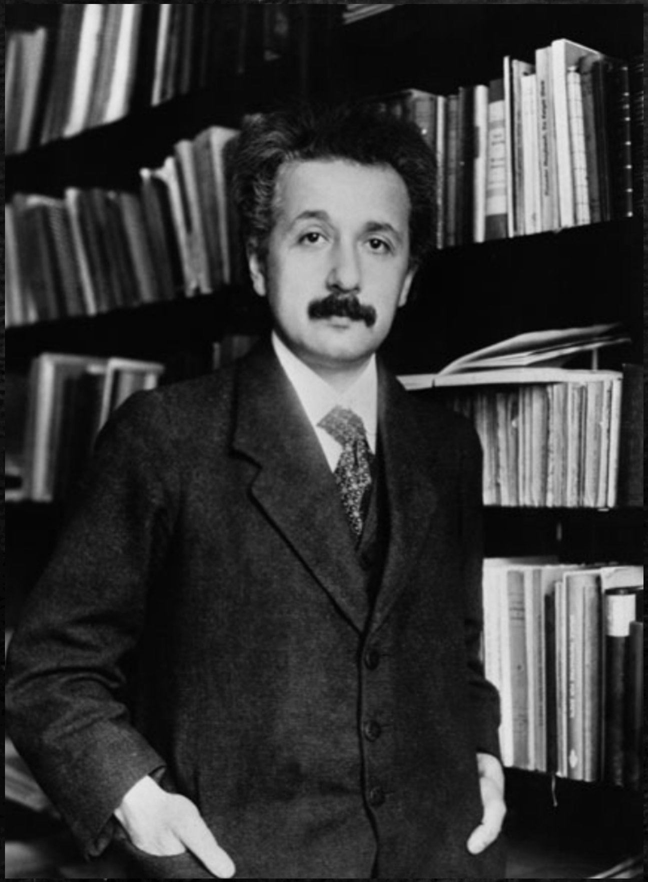




``I cannot, namely, admit that the assertion that the theory of relativity traces physics back to geometry has a clear meaning. [...] The fact that the metric tensor is denoted as ``geometrical'' is simply connected to the fact that this formal structure first appeared in the area of study denoted as ``geometry''. However, this is by no means a justification for denoting as ``geometry'' every area of study in which this formal structure plays a role, not even if for the sake of illustration one makes use of notions which one knows from geometry. Using a similar reasoning Maxwell and Hertz could have denoted the electromagnetic equations of the vaccuum as ``geometrical'' because the geometrical concept of a vector occurs in these equations.'' Einstein (1927), Review of Meyerson. (See Lehmkuhl [2014] for analysis and similar quotes from other decades.)

Marble and Wood

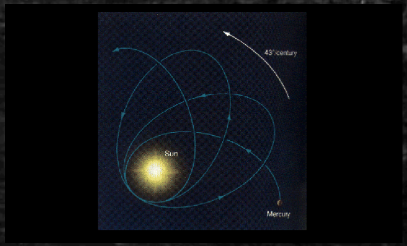




``[GR] is sufficient --- as far as we know --- for the observation of the observed facts of celestial mechanics. But it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low-grade wood (right side of equation). '' Einstein (1936)

``I wonder if the equation $G_{\mu\nu} = T_{\mu\nu}$ still has any reality left within itself, especially when facing quanta. I doubt it, strongly. However, the left hand side surely contains a deeper truth. If the equations $R_{\mu\nu} = 0$ really determine the behaviour of the singularities, then the law governing this behaviour would be rooted in a much deeper reason than the former equation, which is not unified and of only a phenomenogical kind."
Einstein to Besso, 11 August 1926

Why is $T_{\mu\nu}$ `just a phenomenological representation of matter?



- Already in the 1916 review paper on GR, Einstein insisted that GR does not change *anything* about the special relativistic theory of matter; GR incorporates the SR model of matter, but is *not supposed to* tell us anything new about matter.
- Moreover, very different material systems can have the very same energy-momentum tensor, the same mass-energy distribution; just knowing the energy tensor does not tell us whether we have an electromagnetic field or a viscous fluid.
- In short: knowing a $T_{\mu\nu}$ does not tell us the nature of the matter present. It only tells us *one* of its (derivative) properties, and even that in typically a highly idealised fashion.

Two questions now:

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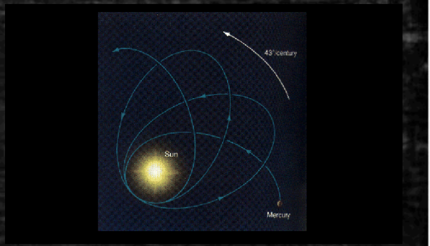
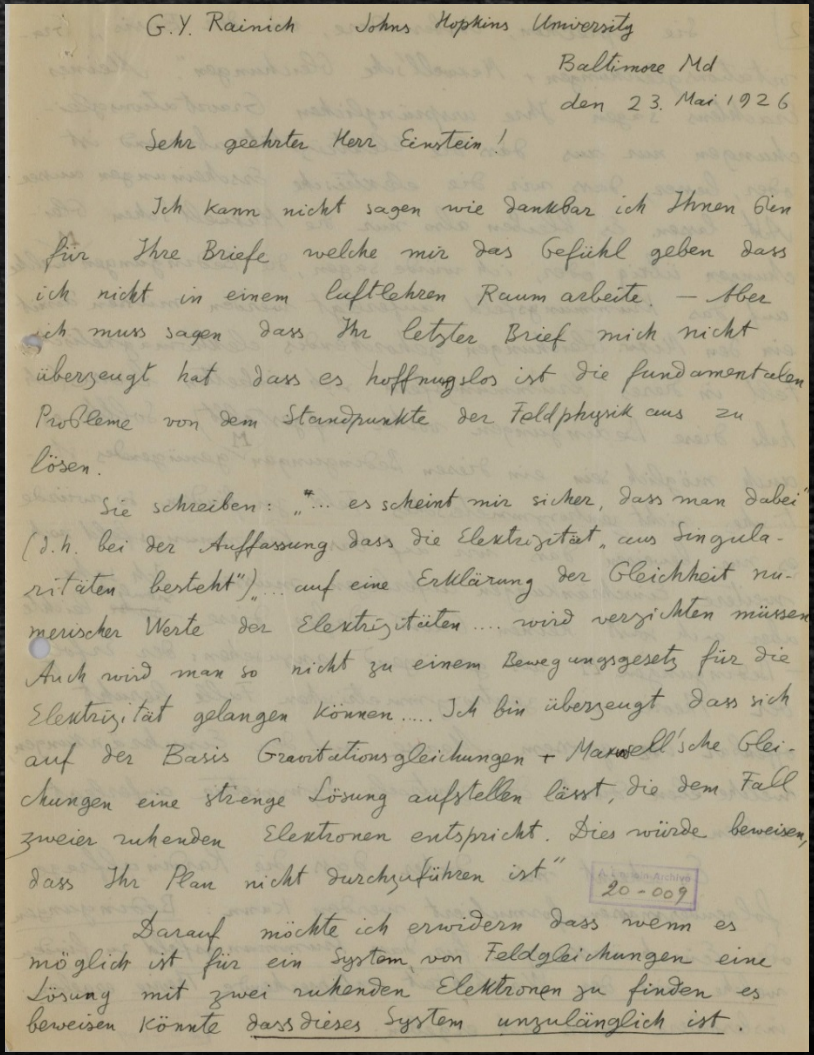


Why does Einstein suddenly allow for singularities?

Einstein's 1926 correspondence with Yuri Rainich shows the genesis of the key ideas in the Einstein-Grommer paper.



G.Y. Rainich Johns Hopkins University
Baltimore Md
den 23. Mai 1926

Sehr geehrter Herr Einstein!

Ich kann nicht sagen wie dankbar ich Ihnen bin für Ihre Briefe welche mir das Gefühl geben dass ich nicht in einem luftleeren Raum arbeite. – Aber ich muss sagen dass Ihr letzter Brief mich nicht überzeugt hat dass es hoffnungslos ist die fundamentalen Probleme von dem Standpunkte der Feldphysik aus zu lösen.

Sie schreiben: „... es scheint mir sicher, dass man dabei" (d.h. bei der Auffassung dass die Elektrizität aus Singularitäten besteht") ... auf eine Erklärung der Gleichheit numerischer Werte der Elektrizitäten .... wird verzichten müssen. Auch wird man so nicht zu einem Bewegungsgesetz für die Elektrizität gelangen können ..... Ich bin überzeugt dass sich auf der Basis Gravitationsgleichungen + Maxwell'sche Gleichungen eine strenge Lösung aufstellen lässt, die dem Fall zweier ruhenden Elektronen entspricht. Dies würde beweisen dass Ihr Plan nicht durchzuführen ist"

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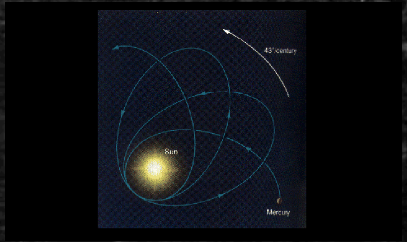
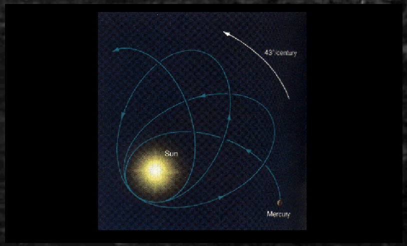
Darauf möchte ich erwidern dass wenn es möglich ist für ein System von Feldgleichungen eine Lösung mit zwei ruhenden Elektronen zu finden es beweisen könnte dass dieses System unzulänglich ist.

Rainich's Physica Note



- On 21 February 1926 Rainich sent Einstein an article he had just submitted to `Physica'. In the paper, Rainich criticizes an argument from a paper Einstein had published in November 1925, entiteld `Electron and Relativity Theory'.
- Einstein had argued that any theory which i.) represents the electromagnetic field by an antisymmetric tensor and ii) which has a solution capable of representing an electron with negative charge *-e* and mass *m*, will also allow for a solution with charge *+e* and mass *m*. Einstein considered every such theory as in contradiction with experience.
- Rainich now argued that the theory would only contradict experiment if it would allow for a solution in which *both* negative and positive electrons existed without moving towards one another.

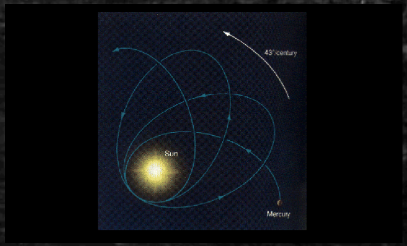


Rainich's Physica Note

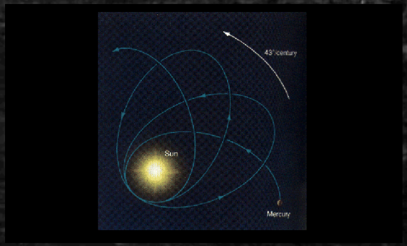
- Rainich stated that in a *linear* theory the existence of a static electron and a static `positron' solution would indeed imply a static solution in which the two coexist without moving toward one another.

- However, in a *non-linear* theory like GR, the existence of such a solution is not implied. In a letter to Einstein from 5 April 1926, Rainich adds that in contrast to a linear theory, in a non-linear theory the field of one particle may heavily constrain the properties the second particle can have.

Einstein's immediate reaction



``I hurry to answer your letter, happy that you struggle with the same questions as I myself have for such a long time, to no avail. The cardinal question is of course whether one should think of electricity as continuous or made up of singularities. The latter option seems easier at first sight, since one could then just stick with the Maxwell equations without adding anything to them. ... [But i]t won't be possible to gain the equations of motion of electricity in this way... . I am convinced that one could find a strict solution on the basis of the gravitational equations + Maxwell equations, which would represent the case of two electrons at rest (as singularities). For the case in which the particles in question have no electric charge this has already been shown by Weyl and Levi-Civita (special case of axial symmetry). This would show that your plan cannot be carried out.'' Einstein to Rainich, 18 April 1926.

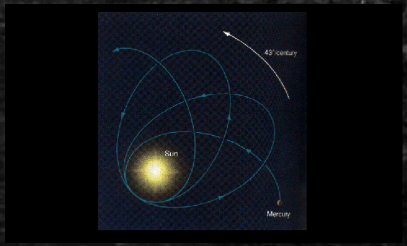


Rainich insists

``I cannot tell you how grateful I am for your letters, which give me the feeling that I am not working in a vaccuum. - But I have to say that your last letter did not convince me… . […] '' Rainich to Einstein, 23 May 1926.

- In what follows, Rainich insists on the points of his previous letter: it is not clear that GR admits a solution that should be interpreted as representing two particles (represented as singularities) at rest with respect to one another.

Between 23 May and 6 June 1926

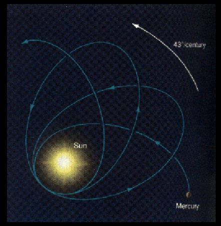


- I conjecture that between Rainich's letter of 23 May and Einstein's answer of 6 June, Einstein must have gone back to the papers by Levi-Civita and Weyl (and Bach) that he had referred to in his previous letter.

- For if he had, and if he had put Rainich's *Physica* note next to Bach's and Weyl's 1922 paper discussing the axisymmetric solution of two approximately spherical bodies, he would have seen their point --- in line with Rainich --- that this two-body solution is *not* the superposition of two static one-particle solutions. Instead, starting from two one-particle solutions, the way in which the particles can co-exist and form a two-particle solution is heavily constrained in a non-linear theory.

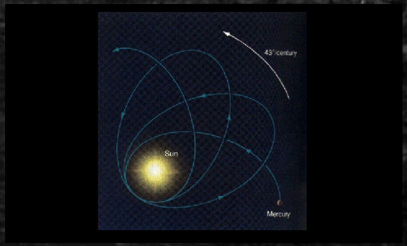
- In particular, in the Bach-Weyl solution there is a singularity along the z-axis between the two particles that is responsible for keeping the particles at rest with respect to one another.

How Einstein repurposes the Bach-Weyl (BW) solution in the Einstein-Grommer paper



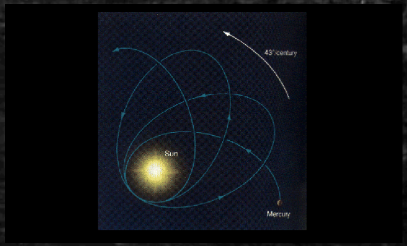
- Weyl's aim had been to find a solution to the vaccuum field equations that could be interpreted as respresenting two gravitating particles at rest with respect to one another. He found that such a solution exists, but that it has a singularity along the z-axis separating the two particles.
- Einstein now implicitly demanded that singularities are only allowed to exist at the points where the particles are located.
- He interprets the BW-solution as representing one particle subject to an external gravitational field. He then asks what has to happen in the Bach-Weyl solution in order to get rid of the singularity along the z-axis.
- He finds that this is only possible if the external gravitational field vanishes at the point where the particle exists.
- Thus, he concludes, in the full, non-linear theory, there is no physical solution of a particle at rest but subject to an external gravitational field.

The jump from the Bach-Weyl solution to the problem of motion



- Einstein concludes that in the full, non-linear theory, there is no physical solution of a particle at rest but subject to an external gravitational field.
- Thus, he says, in GR it follows from the field equations that a particle cannot be at rest when subject to a gravitational field.
- So the field equations predict whether a particle moves; they predict *that* it will move.
- From here it is only a small step to expect the field equations to determine *how* the particle will move.

➢ The problem of motion.



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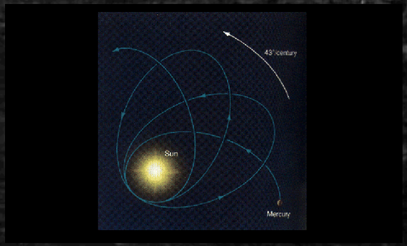
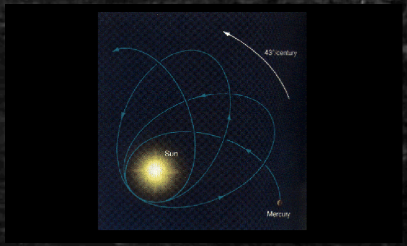
- In what follows, Rainich insists on the points of his previous letter: it is not clear that GR admits a solution that should be interpreted as representing two particles (represented as singularities) at rest with respect to one another.

Einstein to Rainich: the core question of 6 June 1926



``I completely agree with your main point. If a theory has a solution which represents two electrons *at rest*, then it is inadequate. This was indeed the reason why I thought that I had to reject a theory which regards electrons *as singularities*. For I had thought to have seen that any such theory would have solutions with electrons at rest. But it now seems that I was wrong about this. Either way, this is the core question: A theory is sensible only if it allows to derive the equations of motion of particles without any extra assumptions. Whether the electrons are treated as singularities or not does not really matter *in principle.*"
Einstein to Rainich, 6 June 1926 (emphasis in original).

A change in what the “core question” is



- Einstein's core question on 18 April 1926: whether particles are modeled by continuous fields or by singularities.
- Einstein's core question on 6 June 1926: whether the equations of motion of particles (however modeled) follow from the field equations.
- Note how Einstein keeps the admission of singularities in the theoretical description heavily constrained: Only material objects are allowed to be represented by singularities, but singularities outside of matter are not to be admitted.
- Why? Remember Einstein’s view of matter in the context of GR: it is a blind spot of the theory, as a theory only of gravitational fields, a blind spot to be illuminated only by GR’s successor theory.

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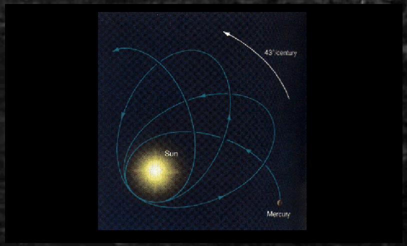
A New Approach: Deriving equations of motion from vaccuum field equations.

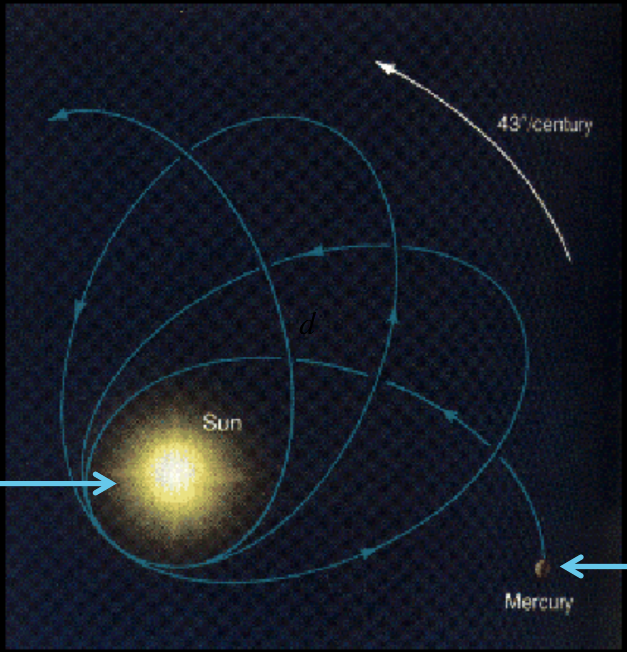


``We are thus led to a third approach which, apart from the gravitational and the electromagnetic field, does not admit any other field variables, … but instead admits singular world lines. … It has turned out that the equations of motion of the singularities are completely determined by the field equations and the character of the singularities.'' Einstein and Grommer 1927.

The new way applied to GR

The Einstein field equations:

$$R_{\mu\nu} = 0$$

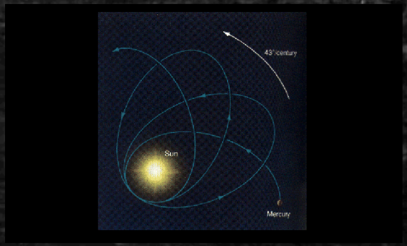


- BUT there is a price to pay: matter is represented by singularities.

The geodesic equation:

$$\frac{d^2 x_\tau}{ds^2} + \Gamma^\tau{}_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0$$

How to do it?



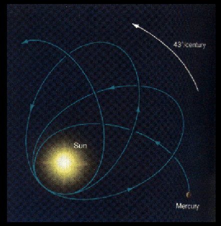
- Reformulate the Einstein field equations as a surface integral.
- Pick a `singular curve' supposed to represent the path of a material particle.
- Obtain an `equilibrium condition' for the energy-pseudo-tensor of the gravitational field around the curve. (Here Einstein repurposes another of Weyl's papers.)
- Approximate the metric field around the singular curve by splitting the total metric into an `inner metric' $\overline{\gamma}_{\mu\nu}$ and an `outer metric' $\overline{\overline{\gamma}}_{\mu\nu}$. Observe that the `outer metric' is entirely regular.
- Integrate the surface integral `around' the curve.

➢ Then it follows that the curve is a geodesic of the outer metric $\overline{\overline{\gamma}}_{\mu\nu}$

Outline

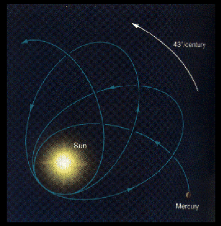
1. Einstein and Grommer on the three ways of relating field equations and equations of motion
2. Does $T_{\mu\nu}$ belong to General Relativity proper?
3. The Einstein-Rainich correspondence: singularities and non-linearity
4. The Einstein-Grommer approach: from vaccuum field equations to equations of motion
5. Reflections on the Einstein-Grommer approach
6. Conclusion

Problems of the approach: Accepting singularities



- `A singularity is not even part of spacetime. How should it be possible to describe its motion *in* said spacetime? It does not make sense!'
- Possible answer 1: Indeed the singularity is not part of spacetime. But Einstein and Grommer do not take it to be part of the theory they are developing; the singularity is a place-holder for something that is not described by GR. Just like representing the Sun by the Schwarzschild solution of the Einstein field equations does not commit us to thinking that there `really' is a singularity at its center, the Einstein-Grommer approach does not commit us to thinking of matter as `really' singular. It's an approximation.

Problems of the approach: Getting rid of singularities; but is it matter?

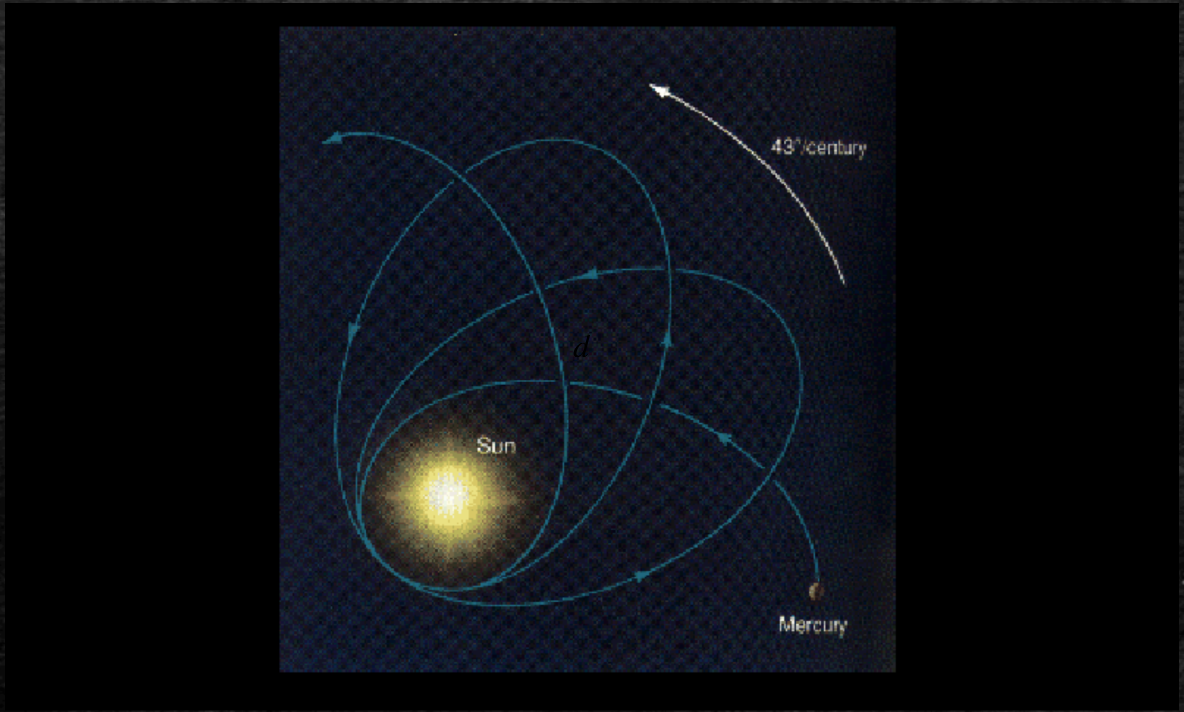


- `A singularity is not even part of spacetime. How should it be possible to describe its motion *in* said spacetime? It does not make sense!'

- Possible answer 2: Even though Einstein and Grommer choose the `inner metric' in such a way that there is a singularity at r=0, it seems that their argument does not depend on this. They could leave the inner metric undetermined and only judge that the curve surrounded and constrained by the specified outer metric is a geodesic of the outer metric.

- This brings about a new problem though: why should said geodesic be interpreted as the path of a material particle?

Possible Solution: Using knowledge external to GR



The astronomers tell us which of the paths around the sun is Mercury's path. We then ask whether the field equations can tell us that the respective path is a geodesic. Answering this question does not need to include a theoretical representation of Mercury itself.

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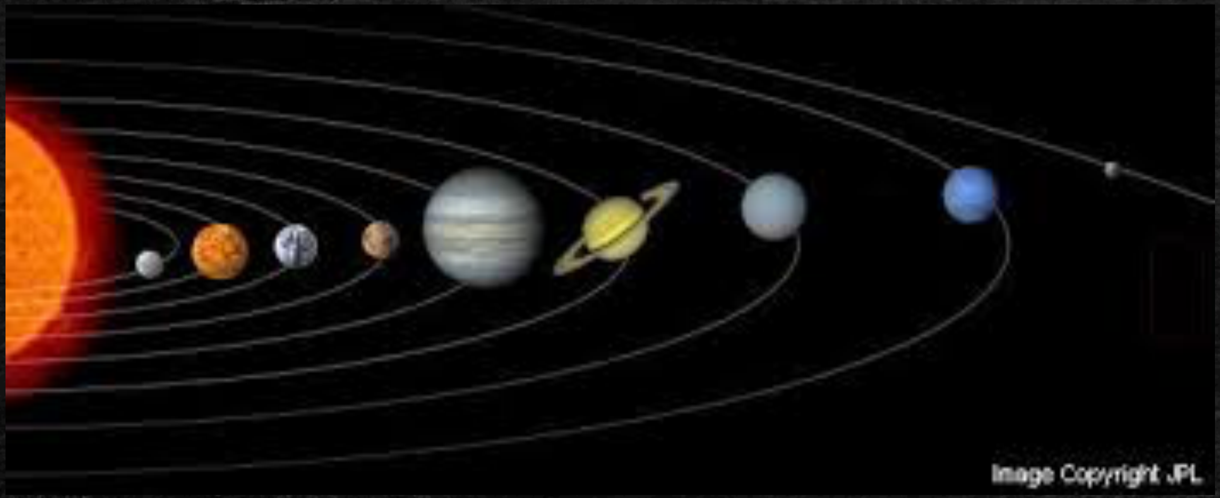

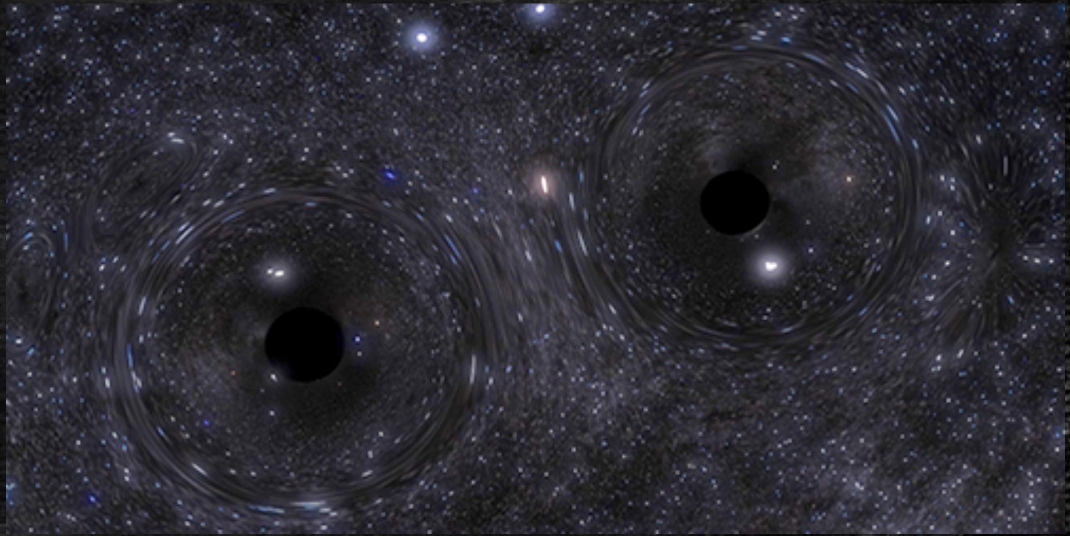

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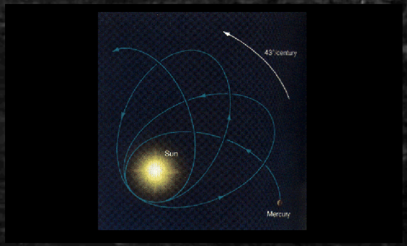

- I started out by reviewing Einstein's and Grommer's three ways in which the field equations and the equations of motion for particles could be related: i) the Newtonian way, ii) the T way, and iii.) by way of using only the vacuum field equations and allowing for singularities.

- I showed why Einstein disliked ii), and how he came to follow iii.) instead. The turning point towards Einstein allowing for singularities to represent material particles took place in Einstein's correspondence with Yuri Rainich, and his reconsidering the Bach-Weyl solution during the correspondence.

- We then looked at Einstein's and Grommer's proof method, and suggested that a) modeling particles as singularities is not as problematic as one might think, and b) that it might be possible to rid the approach of singularities.

Thank you